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On Inclusion Probabilities for Order Sampling

Bengt Rosén

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On Inclusion Probabilities for Order Sampling (*Bengt Rosén*)

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On Inclusion Probabilities for Order Sampling

Bengt Rosén

Abstract.

$\lambda_1, \lambda_2, \ldots, \lambda_N$ denote target inclusions probabilities for a, possibly approximate, πps sampling scheme with (fixed) sample size n from a size N population, and $\pi_1(n), \pi_2(n), \ldots, \pi_N(n)$ the factual inclusion probabilities. Using results for order sampling (Rosén 1997a) the author exhibited (in Rosén 1997b)a class of approximate πps schemes, order πps (OSπps) schemes, and advised allied estimation procedures. These procedures were derived from limit results for linear statistics, and no per se study of the $\pi_i(n)$ was needed. The (asymptotic) πps property of the OSπps schemes was proved on a "macro" level, to the effect that the quasi Horvitz - Thompson estimator with λ_i instead of π_i yields consistent estimation.

The chief result in the present paper is that the asymptotic πps property holds also on "micro" level, i.e. that the individual π_i lies close to the target λ_i, i.a. we show that (1) below holds under general conditions. Special attention is paid to certain particular OSπps schemes: uniform, exponential and Pareto OSπps.

$$\pi_i(n)/\lambda_i \rightarrow 1, \text{ as sample size n (and population size N) tend to infinity}, \; i = 1,2,\ldots,N. \quad (1)$$

As main technical tools we derive error bounds for a generalized version of the approximation $\pi_i(n) \approx \lambda_i$, valid for general order sampling. The error bounds are used to prove the convergence in (1), and also to provide information on convergence rates.

We also consider exact formulas for π_i, a main aim being to show how unmanageable such formulas become when n and N are not very small. In spite of this, manageable formulas are derived for a specific, non - trivial OSπps situation. They are used in a numerical study of the goodness of the approximation $\pi_i \approx \lambda_i$. The findings indicate that Pareto πps yields best approximation, and suggest the following rule of thumb: For Pareto OSπps, $\pi_i(n)$ differs only negligibly from λ_i if $\min(n, N-n) \geq 5$.

CONTENTS

On Inclusion Probabilities for Order Sampling

1 Basic notions and outline of the paper

A means for utilizing auxiliary information in sample surveys is πps sampling, i.e. sampling (without replacement) with inclusion probabilities proportional to given size measures. Rosén (1997b) introduced a novel class of πps schemes, called order πps schemes, and advised procedures for point and variance estimation for such schemes. These procedures are based on results for a more general notion called "order sampling" introduced in Rosén (1997a). We start by reviewing the definitions of the sampling schemes.

1.1 Definition of order sampling, notably order πps sampling

The following definition is structured by successive specialization: Order sampling - order sampling with fixed distribution shape - order πps sampling.

DEFINITION 1.1 : To each unit k in a population $U=(1,2,...,N)$ is associated a probability distribution $F_k(t)$ with density $f_k(t)$, $0 \le t < \infty$. A sample size n, $n \le N$, is prescribed.

a. ***Order sampling*** with ***sample size*** n and ***order distributions*** $\mathbf{F}=(F_1,F_2,...,F_N)$ is carried out as follows. Independent ***ranking variables*** $Q_1, Q_2, ..., Q_N$ with distributions $F_1, F_2, ..., F_N$ are realized. The units with the ***n smallest*** Q - values constitute the sample. The scheme is referred to by $OS(n;\mathbf{F})$.

b. H(t) is a probability distribution with density h(t), $0 \le t < \infty$, and $\underline{\theta}=(\theta_1,\theta_2,...,\theta_N)$ are given positive real numbers. ***Order sampling*** with ***fixed shape distribution*** H, ***intensities*** $\underline{\theta}$ and ***sample size*** n is $OS(n;\mathbf{F})$ with the following order distributions;

$$F_k(t)=H(t\cdot\theta_k), \quad \text{with density} \quad f_k(t)=\theta_k\cdot h(t\cdot\theta_k), \quad 0\le t<\infty, \quad k=1,2,\dots,N. \quad (1.1)$$

The general scheme is referred to by $OSFS(n;H;\underline{\theta})$. Particular schemes are named by their shape distributions.

c. $\underline{\lambda}=(\lambda_1,..,\lambda_N)$ are given real numbers which satisfy;

$$0<\lambda_k<1, \quad k=1,2,\dots,N, \quad \sum_{k=1}^{N}\lambda_k=n. \quad (1.2)$$

Order πps sampling with ***sample size*** n, ***shape distribution*** H and ***target inclusion probabilities*** $\underline{\lambda}$ is $OSFS(n;H;\underline{\theta})$ with intensities;

$$\theta_k=H^{-1}(\lambda_k), \quad k=1,2,\dots,N, \quad (H^{-1} \text{ denotes inverse function}). \quad (1.3)$$

The general order πps scheme is referred to by $OS\pi ps(n;H;\underline{\lambda})$. Particular OSπps schemes are named by their shape distributions.

We will pay special attention to schemes with the following shape distributions.

Uniform OSFS: $H(t)=t,\ h(t)=1,\ 0\le t\le 1,\ H(t)=1,\ h(t)=0,\ 1\le t<\infty.$ (1.4)

The corresponding OSπps scheme, ***uniform OSπps***, has intensities;

$$\theta_k=H^{-1}(\lambda_k)=\lambda_k, \quad k=1,2,\dots,N. \quad (1.5)$$

Exponential OSFS: $H(t)=1-e^{-t},\quad h(t)=e^{-t},\ 0\le t<\infty.$ (1.6)

The corresponding OSπps scheme, ***exponential OSπps***, has intensities;

$$\theta_k=H^{-1}(\lambda_k)=-\log(1-\lambda_k), \quad k=1,2,\dots,N. \quad (1.7)$$

Pareto OSFS: $H(t)=t/(1+t),\quad h(t)=1/(1+t)^2,\ 0\le t<\infty.$ (1.8)

The corresponding OSπps scheme, ***Pareto OSπps***, has intensities;

$$\theta_k=H^{-1}(\lambda_k)=\lambda_k/(1-\lambda_k), \quad k=1,2,\dots,N. \quad (1.9)$$

1.2 Outline and main results

In the sequel π_i has its usual sampling theory meaning;

$$\pi_i(n) = \text{inclusion probability for unit } i \text{ in a sample of size n}, \quad i = 1,2,...,N. \tag{1.10}$$

The point and variance estimation procedures in Rosén (1997 a & b) are derived in a somewhat non - standard way. Instead of "usual" estimators (of Horvitz - Thompson and Sen - Yates - Grundy type) based on known inclusion probabilities, the estimators are deduced from limit results for linear statistics. This approach circumvents the obstacle that numerically manageable formulas for inclusion probabilities are unfeasible for order sampling. In the limit approach no per se study of inclusion probabilities is needed. Matters work the other way round, the estimators yield conjectures about approximate π_i - values, as stated below.

APPROXIMATE INCLUSION PROBABILITIES: Consider general order sampling OS(n; **F**), and let ξ be determined by the relation;

$$\sum_{k=1}^{N} F_k(\xi) = n. \tag{1.11}$$

Then the following approximation works well under general conditions;

$$\pi_i(n) \approx F_i(\xi), \quad i = 1,2,...,N. \tag{1.12}$$

For OSπps(n; H; $\underline{\lambda}$), (1.12) specializes, for any shape distribution H, to;

$$\pi_i(n) \approx \lambda_i, \quad i = 1,2,\ldots,N. \tag{1.13}$$

Remark 1.1: Details on the specialization of (1.12) to (1.13) for OSπps are given in Section 3 Rosén (1997 b), in particular the following technical result, which will be useful later on.

For an OSπps scheme, the solution to equation (1.11) is $\xi = 1$. ¤ (1.14)

In Rosén (1997a & b) is shown that asymptotic correctness of (1.12) and (1.13) holds on "macro" level, in the sense that the "quasi" Horvitz - Thompson estimator with λ_i instead of π_i yields consistent estimation. A chief aim in the present paper is to provide justification also on "micro" level, i.e. that individual π_i lie close to the corresponding target value λ_i. In other words, we justify that OSπps schemes asymptotically live up to their πps name. In Section 4 is proved that, under general conditions, (1.13) holds for the OSπps schemes of particular interest, uniform, exponential and Pareto OSπps, to the following effect.

$$\pi_i(n)/\lambda_i \to 1, \quad \text{as the sample size n tends to infinity}, \quad i = 1,2,...,N, \tag{1.15}$$

where the limit frame - work is the customary one in finite population sampling contexts, a sequence of populations whose sizes of tend to infinity is considered.

The main technical tools in the proofs are theoretical error bounds for the approximations (1.12) and (1.13). Even if OSπps lies in our focus of interest, the derivation of error bounds is most comprehensible when made for general order sampling, which is the task in Section 2. In Section 3 the bounds are specialized to the OSπps schemes of particular interest.

The error bounds also yield results about the rate of convergence in (1.15). These results furnish more direct proofs of the fact shown in Rosén (1997 a & b) that the point estimator yields consistent estimation, i.e. that relative estimation errors are asymptotically negligible. They also provide information on the rate of convergence to 0 for estimator biases. In Section 4 we also discuss the following matter. The approach in this paper does presumably not lead to best possible orders of magnitude for the error bounds. However, derivation of sharper bounds would increases the complexity of the problem dramatically.

Even if limit results of type (1.15) have theoretical interest, they are less interesting from a practical point of view. Then one wants to know if the approximation (1.13) works sufficiently well in a specific finite situations. The theoretical error bounds are, as usual, not sharp enough to yield practically interesting information. In Section 5 we consider exact formulas for π_i. The chief aim is, however, not to exhibit formulas for practical use, rather to show how unmanageable such formulas become when n and N are not very small. In spite of this, reasonably manageable formulas are derived for a specific, but non - trivial and fairly general, OSπps situation. These formulas are used in a numerical study of the goodness of the approximation (1.13). Even if we cannot draw comprehensive conclusions from the numerical findings, we mean that they quite strongly indicate the following result.

For Pareto OSπps, π_i differs only negligibly from λ_i if $\min(n, N-n) \geq 5$. (1.16)

The findings also indicate that π_i converges quite rapidly to λ_i for uniform and exponential OSπps as well, but Pareto πps is the scheme with best approximation.

Notation used throughout the paper: P, E, V and $D = \sqrt{V}$ denote probability, expectation, variance and standard deviation. $\mathbf{1}(C)$ stands for the indicator of the set/event C, and $\#C$ for the number of elements in it. The natural logarithm is denoted by log.

2 Error bounds for general order sampling

2.1 The basic estimate

We start with a notation.

DEFINITION 2.1: For a (fixed) population unit i, set;

$$G_i(t) = \sum_{k \neq i} F_k(t), \quad 0 \leq t < \infty, \quad i = 1, 2, \ldots, N. \tag{2.1}$$

Then, for $0 \leq y \leq N-1$,

$$\chi_i(y) \text{ is the solution to the equation (in t): } G_i(t) = y, \ i = 1, 2, \ldots, N, \tag{2.2}$$

The following lemma will be basic in our study of the approximation error in (1.3).

LEMMA 2.1: Consider OS(n; **F**) from U. $F_i(\xi)$ and χ_i, $i \in U$, are specified by (1.11) and (2.2). Presume that the following conditions are met for some $b \geq 1$, some γ_i and some ρ_i.

$$f_i(t) \leq \gamma_i, \quad \chi_i(n-b) \leq t \leq \chi_i(n+b), \tag{2.3}$$

$$\sum_{k \neq i} f_k(t) \geq \rho_i, \quad \chi_i(n-b) \leq t \leq \chi_i(n+b), \tag{2.4}$$

$$\sum_{k \neq i} F_k(t) \cdot (1 - F_k(t)) \geq 1, \quad \chi_i(n-b) \leq t \leq \chi_i(n+b). \tag{2.5}$$

Then the inequality below holds;

$$|\pi_i - F_i(\xi)| \leq (b+1) \cdot \frac{\gamma_i}{\rho_i} + 2.1 \cdot e^{-b/\sqrt{n+b}}, \quad i = 1, 2, \ldots, N. \tag{2.6}$$

Proof: We regard n and $b \geq 1$ as fixed, and introduce the following short notation;

$$\omega_i = \chi_i(n-b) \quad \text{and} \quad \psi_i = \chi_i(n+b). \tag{2.7}$$

As in Definition 1.1, $Q_1, Q_2, \ldots, Q_N$ denote the ranking variables. Introduce the variables A_i and B_i, which count the number of Q:s with values less than ω_i and ψ_i respectively;

$$A_i(n; b) = \#\{k : k \neq i, Q_k \leq \omega_i\}, \quad B_i(n; b) = \#\{k : k \neq i, Q_k \leq \psi_i\}. \tag{2.8}$$

Since the sample consists of the units with the n smallest outcomes of $Q_1, Q_2, \ldots, Q_N$;

$$\mathbf{1}(\text{unit } i \text{ is included in the sample}) \geq \mathbf{1}(A_i < n) \cdot \mathbf{1}(Q_i \leq \omega_i), \tag{2.9}$$

$$\mathbf{1}(\text{unit } i \text{ is not included in the sample}) \geq \mathbf{1}(B_i \geq n) \cdot \mathbf{1}(Q_i > \psi_i). \tag{2.10}$$

The independence of the Q:s yields that the events to the right in (2.9) are independent. The same holds for (2.10). By using this and taking expectation we get;

$$\pi_i \geq P(A_i < n) \cdot P(Q_i \leq \omega_i) = (1 - P(A_i \geq n) \cdot F_i(\omega_i) \geq F_i(\omega_i) - P(A_i \geq n), \tag{2.11}$$

$$1 - \pi_i \geq P(B_i \geq n) \cdot P(Q_i > \psi_i) = (1 - P(B_i < n)) \cdot (1 - F_i(\psi_i)) \geq 1 - F_i(\psi_i) - P(B_i < n). \tag{2.12}$$

From (2.11) and (2.12) follows;

$$-[F_i(\xi) - F_i(\omega_i)] - P(A_i \geq n) \leq \pi_i - F_i(\xi) \leq F_i(\psi_i) - F_i(\xi) + P(B_i < n). \tag{2.13}$$

With t interpreted as "time", $\chi_i(y)$ is the time when $G_i(t)$ crosses the level y. In the same vein, ξ in (1.11) is the time when $G_i(t)$ crosses the level $n - F_i(\xi)$. This together with the fact that $G_i(t)$ is non-decreasing as t increases and $0 \leq F_i(\xi) \leq 1$ yields;

$$\text{For } b \geq 1: \quad \omega_i \leq \xi \leq \psi_i, \quad i = 1, 2, \ldots, N. \tag{2.14}$$

Now (2.13) and (2.14) imply;

$$|\pi_i - F_i(\xi)| \leq \max\{F_i(\xi) - F_i(\omega_i), F_i(\psi_i) - F_i(\xi)\} + \max\{P(A_i \geq n), P(B_i < n)\}. \tag{2.15}$$

To pursue the estimate (2.15) we start by estimating $\max\{ F_i(\xi) - F_i(\omega_i), F_i(\psi_i) - F_i(\xi)\}$. The mean value theorem tells that $F_i(t) - F_i(\xi) = (t - \xi) \cdot f_i(\theta)$ for a θ between t and ξ. This together with (2.3) yields;

$$F_i(\xi) - F_i(\omega_i) \leq (\xi - \omega_i) \cdot \gamma_i \quad \text{and} \quad F_i(\psi_i) - F_i(\xi) \leq (\psi_i - \xi) \cdot \gamma_i, \quad i = 1, 2, \ldots, N. \tag{2.16}$$

We employ the following estimate in (2.16), with G_i as in (2.1) and G' denoting derivative;

$$x - y \leq [G_i(x) - G_i(y)] / \inf_{x \leq t \leq y} G_i'(t), \quad x < y. \tag{2.17}$$

By (2.1) and (2.7), $G_i(\xi) = n - F_i(\xi)$ and $G_i(\omega_i) = n - b$. Moreover, (2.4) states that $G_i'(t)$ is $\geq \rho_i$ on the interval $[\omega_i, \psi_i]$. Hence (2.17) yields;

$$\xi - \omega_i \leq [n - F_i(\xi) - (n - b)] / \rho_i \leq b / \rho_i. \tag{2.18}$$

By analogous arguments follows;

$$\psi_i - \xi \leq [(n + b) - (n - F_i(\xi))] / \rho_i \leq (b + 1) / \rho_i. \tag{2.19}$$

The estimates (2.16), (2.18) and (2.19) yield;

$$\max\{ F_i(\xi) - F_i(\omega_k), F_i(\psi_k) - F_i(\xi)\} \leq (b + 1) \cdot \gamma_i / \rho_i. \tag{2.20}$$

Next we estimate the tail probabilities $P(A_i \geq n)$ and $P(B_i < n)$. A_i in (2.8) can be viewed as a sum of independent Bernoulli variables $X_k = \mathbf{1}(Q_k \leq \omega_i)$, $k \neq i$, with means $E[X_k] = F_k(\omega_i)$;

$$A_i = \sum_{k \neq i} X_k = \sum_{k \neq i} \mathbf{1}(Q_k \leq \omega_i). \tag{2.21}$$

We use the "exponential bound" estimate in Lemma 2.2 below. First, by (2.21), standard formulas for Bernoulli variables, (2.7), (2.1) and (2.2);

$$E(A_i) = \sum_{k \neq i} F_k(\omega_i) = n - b, \tag{2.22}$$

$$V(A_i) = D^2(A_i) = \sum_{k \neq i} F_k(\omega_i) \cdot (1 - F_k(\omega_i)) \leq \sum_{k \neq i} F_k(\omega_i) = n - b \leq n. \tag{2.23}$$

Lemma 2.2 together with (2.21), (2.22) and (2.23) yield (2.24) below. Note that (2.5) and (2.23) imply that $1 \leq D(A_i) \leq \sqrt{n}$;

$$P(A_i \geq n) = P(A_i \geq E(A_i) + b) = P(A_i \geq E(A_i) + [b / D(A_i)] \cdot D(A_i)) \leq 2.1 \cdot e^{-b/\sqrt{n}}. \tag{2.24}$$

Quite analogously, the relations

$$B_i = \sum_{k\neq i} \mathbf{1}(Q_k \geq \psi_i), \tag{2.25}$$

$$E(B_i) = \sum_{k\neq i} F_k(\psi_i) = n+b, \quad V(B_i) = \sum_{k\neq i} F_k(\psi_i)\cdot(1-F_k(\psi_i)) \leq \sum_{k\neq i} F_k(\psi_i) \leq n+b, \tag{2.26}$$

together with a straightforward modification of Lemma 2.2 yield (2.27) below. Note that (2.5) and (2.26) imply that $1 \leq D(B_i) \leq \sqrt{n+b}$.

$$P(B_i < n) = P(B_i < E(B_i) - b) = P(B_i < E(B_i) - [b/D(B_i)]\cdot D(B_i)) \leq 2.1 \cdot e^{-b/\sqrt{n+b}}. \tag{2.27}$$

Now (2.24) and (2.27) imply;

$$\max\{P(A_i \geq n), P(B_i < n)\} \leq 2.1 \cdot e^{-b/\sqrt{n+b}}. \tag{2.28}$$

Insertion of the estimates (2.20) and (2.28) into (2.15) yields (2.6). ¤

We conclude by deriving the exponential bound inequality that was used above. The result in following lemma goes back on Kolmogorov. For completeness and for exhibition of explicit constants we give a proof, though, which in essence is that of Proposition (i) in Section 18.1 in Loève (1955). Recall that D denotes standard deviation.

LEMMA 2.2: Let $S = X_1+X_2+\ldots+X_m$ be a sum of independent Bernoulli variables. Provided that $D(S) \geq 1$ we have;

$$P(S \geq E(S) + \lambda\cdot D(S)) \leq 2.1\cdot e^{-\lambda}, \quad \lambda \geq 0. \tag{2.29}$$

Proof: By the relation $\mathbf{1}(S \geq s) \leq \exp\{\alpha\cdot(S-s)\}$, $\alpha \geq 0$, we have;

$$\mathbf{1}(S \geq E(S)+\lambda\cdot D(S)) \leq \exp\{\alpha\cdot[S - (E(S)+\lambda\cdot D(S)]\}. \tag{2.30}$$

By taking expectation in (2.30) we get, with $p_k = E(X_k)$;

$$P(S \geq E(S) + \lambda\cdot D(S)) \leq e^{-\lambda\cdot\alpha\cdot D(S)}\cdot E[e^{\alpha\cdot(S-E(S))}] = e^{-\lambda\cdot\alpha\cdot D(S)}\cdot \prod_{k=1}^{m} E[e^{\alpha\cdot(X_k-p_k)}]. \tag{2.31}$$

By expanding in power series and taking expectation we get;

$$E[e^{\alpha\cdot(X_k-p_k)}] = 1 + \frac{\alpha^2}{2!}\cdot E(X_k-p_k)^2 + \frac{\alpha^3}{3!}\cdot E(X_k-p_k)^3 + \frac{\alpha^4}{4!}\cdot E(X_k-p_k)^4 + \ldots \tag{2.32}$$

Since $|X_k - p_k| \leq 1$ we have: $|E(X_k - p_k)^u| \leq E|X_k - p_k|^u \leq E(X_k - p_k)^2 = V(X_k)$, $u=2,3,4,\ldots$. By using these estimates in (2.32) together with $1+x \leq e^x$, $x \geq 0$, we get, provided that $|\alpha| \leq 1$;

$$\begin{aligned} E[e^{\alpha\cdot(X_k-p_k)}] &\leq 1 + \frac{\alpha^2\cdot V(X_k)}{2}\cdot[1+\frac{\alpha}{3}+\frac{\alpha^2}{3\cdot 4}+\ldots] \leq 1+0.72\cdot\alpha^2\cdot V(X_k) \leq \\ &\leq \exp\{0.72\cdot\alpha^2\cdot V(X_k)\}. \end{aligned} \tag{2.33}$$

By employing (2.33) in (2.31) and noting that $\Sigma_k V(X_k) = D(S)^2$ we get;

$$P(S \geq E(S) + \lambda\cdot D(S)) \leq \exp\{-\lambda\cdot\alpha\cdot D(S) + 0.72\cdot\alpha^2\cdot D(S)^2\}. \tag{2.34}$$

Now set $\alpha = 1/D(S)$ in (2.34) and note that $D(S) \geq 1$ implies $\alpha \leq 1$;

$$P(S \geq E(S) + \lambda\cdot D(S)) \leq \exp\{-\lambda + 0.72\} \leq 2.1\cdot e^{-\lambda}. \tag{2.35}$$

Thereby the lemma is proved. ¤

2.2 More explicit versions of the error bound

In Lemma 2.1, b is an optional parameter which remains to be chosen. The choice is a bit involved, though, since b enters in the bound (2.6) as well as in conditions (2.3) - (2.5). To untangle the situation we start with an auxiliary algebraic result.

LEMMA 2.3: Presume that $\rho/\gamma > e$. Let b be the positive root to the equation;

$$x^2 = (n+x)\cdot[\log(\rho/\gamma)]^2. \tag{2.36}$$

Then;

$$\text{(i)} \quad \sqrt{n} < \sqrt{n}\cdot\log(\rho/\gamma) \le b \le \sqrt{n}\cdot\log(\rho/\gamma) + [\log(\rho/\gamma)]^2, \tag{2.37}$$

$$\text{(ii)} \quad (b+1)\cdot\frac{\gamma}{\rho} + 2.1\cdot e^{-b/\sqrt{n+b}} \le \frac{\gamma}{\rho}\cdot(\sqrt{n}\cdot\log(\rho/\gamma) + [\log(\rho/\gamma)]^2 + 3.1). \tag{2.38}$$

Proof: Set $g(x) = x^2 - x\cdot[\log(\rho/\gamma)]^2 - n\cdot[\log(\rho/\gamma)]^2$. Check that $g(\sqrt{n}\cdot\log(\rho/\gamma) + [\log(\rho/\gamma)]^2) > 0$ and $g(\sqrt{n}\cdot\log(\rho/\gamma)) < 0$. This yields (2.37). Since b satisfies (2.36) we have;

$$e^{-b/\sqrt{n+b}} = e^{-\log(\rho/\gamma)} = \gamma/\rho. \tag{2.39}$$

Insertion of (2.39) and the right hand side estimate in (2.37) into the left hand side of (2.38) yields the right hand side of (2.38). ¤

THEOREM 2.1: Consider OS(n;F) from a population in which *i* is a specific unit. Let $F_i(\xi)$ and χ_i be according to (1.11) and (2.2). Moreover, let γ_i, ρ_i, ω_i and ψ_i be quantities such that $\rho_i/\gamma_i > e$, and such that the following conditions are met;

$$f_i(t) \le \gamma_i, \quad \text{for } 0 \le t < \infty, \tag{2.40}$$

$$\sum_{k\ne i} f_k(t) \ge \rho_i > 0, \quad 0 \le t \le \psi_i, \tag{2.41}$$

$$\psi_i \ge \chi_i(n + \sqrt{n}\cdot\log(\rho_i/\gamma_i) + [\log(\rho_i/\gamma_i)]^2), \tag{2.42}$$

$$\omega_i \le \chi_i(n - (\sqrt{n}\cdot\log(\rho_i/\gamma_i) + [\log(\rho_i/\gamma_i)]^2)), \tag{2.43}$$

$$\sum_{k\ne i} F_k(t)\cdot(1 - F_k(t) \ge 1, \quad \omega_i \le t \le \psi_i. \tag{2.44}$$

Then;

$$|\pi_i - F_i(\xi)| \le \frac{\gamma_i}{\rho_i}\cdot(\sqrt{n}\cdot\log(\rho_i/\gamma_i) + [\log(\rho_i/\gamma_i)]^2 + 3.1). \tag{2.45}$$

Proof: The claim (2.45) is obtained by applying Lemma 2.1 with b as stated in Lemma 2.3 and by using the estimate (2.38). Note that the left inequality in (2.37) tells that $b \ge 1$. ¤

Next we formulate a version of Theorem 2.1 with more easily checked conditions. However, it requires additional assumptions, notably that the order distribution densities are decreasing.

THEOREM 2.2: Consider OS(n;F) from a population in which *i* is a specific unit. Let ξ and $F_i(\xi)$ be according to (1.11). Presume that;

$$\text{(i)} \quad f_k(t),\ 0 \le t < \infty, \text{ is non-increasing}, \quad k = 1,2,\ldots,N, \tag{2.46}$$

$$f_i(0) \le \gamma_i. \tag{2.47}$$

Set;

$$\kappa = \sum_{k=1}^{N} f_k(0). \tag{2.48}$$

Let ψ_i and ω_i be quantities such that the following conditions are met;

$$\text{(ii)} \quad \sum_{k=1}^{N} F_k(\psi_i) \ge n + \sqrt{n}\cdot\log(\kappa/\gamma_i) + [\log(\kappa/\gamma_i)]^2 + 1. \tag{2.49}$$

$$\text{(iii)} \quad \omega_i \le \xi - \left(\sqrt{n}\cdot\log(\kappa/\gamma_i) + [\log(\kappa/\gamma_i)]^2\right) \Big/ \sum_{k=1}^{N} f_k(\xi). \tag{2.50}$$

(iv) $\sum_{k=1}^{N} F_k(t) \cdot (1 - F_k(t) \geq 1.25, \quad \omega_i \leq t \leq \psi_i.$ (2.51)

Let ρ_i satisfy;

$$\rho_i \leq \sum_{k=1}^{N} f_k(\psi_i) - \max\{f_1(0), f_2(0), \ldots, f_N(0)\}. \tag{2.52}$$

Then, provided that $\kappa/\gamma_i > e$ we have;

$$|\pi_i - F_i(\xi)| \leq \frac{\gamma_i}{\rho_i} \cdot (\sqrt{n} \cdot \log(\kappa/\gamma_i) + [\log(\kappa/\gamma_i)]^2 + 3.1). \tag{2.53}$$

Proof: In the first round we disregard (2.48), and interpret κ in (2.49), (2.50) and (2.53) as a synonym to ρ_i in (2.52). Under this premise we show that (i) - (iv) imply (2.40) - (2.44). Then (2.53) follows from (2.45).

It is readily seen that (2.46) and (2.47) imply (2.40), and (2.46) and (2.52) imply (2.41). Set;

$$G(t) = \sum_{k=1}^{N} F_k(t), \quad 0 \leq t < \infty. \tag{2.54}$$

For $G_i(t)$ in (2.1) we have $G_i(t) \geq G(t) - 1$, Thus (2.47) (with $\kappa = \rho_i$) implies (2.42). To show that (2.50) (with $\kappa = \rho_i$) implies (2.43) note that, by (2.46), the derivative $G'(t)$ is non-increasing. Hence $G(t)$ is convex (upwards). Thus the curve $y = G(t)$ lies under any of its tangents. By drawing the tangent in $t = \xi$ and paying regard to (1.11), which says that $G(\xi) = n$, we get;

$$G(t) \leq n + (t - \xi) \cdot G'(\xi) = n + (t - \xi) \cdot \sum_{k=1}^{N} f_k(\xi), \quad 0 \leq t < \infty. \tag{2.55}$$

(2.55) in combination with $G(t) \geq G_i(t)$ yields;

$$\chi_i(n - (\sqrt{n} \cdot \log(\rho_i/\gamma_i) + [\log(\rho_i/\gamma_i)]^2)) \geq$$
$$\geq \xi - (\sqrt{n} \cdot \log(\rho_i/\gamma_i) + [\log(\rho_i/\gamma_i)]^2) \Big/ \sum_{k=1}^{N} f_k(\xi). \tag{2.56}$$

Now (2.56) implies that (2.43) is satisfied. Finally, (2.44) follows from (2.51) together with the observation that $F_k(t) \cdot (1 - F_k(t)) \leq 0.25$, $k = 1, 2, \ldots, N$. Hence the result is proved for $\kappa = \rho_i$.

Next we take the actual κ into account, and we start with the following observation. If the theorem with (2.48) omitted holds for $\kappa = \kappa_0$, it holds for any $\kappa > \kappa_0$, since (2.49) - (2.51) become more restrictive and the bound (2.52) more generous if κ is increased. So far the theorem has been proved for $\kappa = \rho_i$. Under (2.46) holds, as is readily checked, that κ in (2.48) is greater than ρ_i in (2.52). Hence, the theorem is true in the given formulation. ¤

3 Error bounds for OSπps inclusion probabilities

3.1 Error bounds for OSπps schemes with decreasing shape density

Here we specialize Theorem 2.2 to OSπps schemes. To make the result easy to apply in particular situations we give it an algorithmic structure.

THEOREM 3.1: Consider OSπps$(n; H; \underline{\lambda})$ from a population, in which *i* is a specific unit. Presume that, where h as usual denotes the density of H;

$$h(t), \; 0 \leq t < \infty, \text{ is non-increasing.} \tag{3.1}$$

Step 1: With intensities according to (1.3), i.e.

$$\theta_k = H^{-1}(\lambda_k), \quad k = 1, 2, \ldots, N; \tag{3.2}$$

set;

$$\beta_i = \log(\sum_{k=1}^{N} \theta_k/\theta_i), \tag{3.3}$$

and determine an α_i such that;

$$\alpha_i \geq \beta_i, \tag{3.4}$$

Step 2: Determine a ψ_i so that the following inequality is satisfied;

$$\sum_{k=1}^{N} H(\theta_k \cdot \psi_i) \geq n + \sqrt{n} \cdot \alpha_i + \alpha_i^2 + 1. \tag{3.5}$$

Step 3: With

$$\delta = \sum_{k=1}^{N} \theta_k \cdot h(\theta_k), \tag{3.6}$$

determine an ω_i so that the following inequality is satisfied;

$$\omega_i \leq 1 - (\sqrt{n} \cdot \alpha_i + \alpha_i^2)/\delta. \tag{3.7}$$

Then,

Step 4: provided that the following condition is met;

$$\sum_{k=1}^{N} H(\theta_k \cdot t) \cdot [1 - H(\theta_k \cdot t)] \geq 1.25 \quad , \quad \omega_i \leq t \leq \psi_i, \tag{3.8}$$

Step 5: and ρ_i satisfies;

$$\rho_i \leq \sum_{k=1}^{N} \theta_k \cdot h(\theta_k \cdot \psi_i) - h(0) \cdot \max\{\theta_k\colon\ k = 1, 2, \ldots, N\}, \tag{3.9}$$

Step 6: the following error bound holds;

$$|\pi_i - \lambda_i| \leq \frac{\theta_i \cdot h(0)}{\rho_i} (\sqrt{n} \cdot \alpha_i + \alpha_i^2 + 3.1). \tag{3.10}$$

Proof: The result will be derived from Th. 2.2. The relation $f_i(t) = \theta_i \cdot h(t \cdot \theta_i)$ in (1.1) together with (3.1) implies (2.46) and (2.47) with $\gamma_i = \theta_i \cdot h(0)$. β_i in (3.3) takes the value $\beta_i = \log(\kappa/\gamma_i)$. The relation $F_i(t) = H(t \cdot \theta_i)$ in (1.1) yields that (3.5) is a version of (2.49), and that (2.51) specializes to (3.8). By (1.14), which states that $\xi = 1$, and $f_i(t) = \theta_i \cdot h(t \cdot \theta_i)$ is seen that (3.7) implies (2.50). Likewise, (3.9) implies (2.52). Finally, (3.10) is the present version of (2.53). ¤

In the following we apply Theorem 3.1 to the OSπps schemes that are specified by (1.4)-(1.9). Our main aim is to exhibit background results for the limit considerations in Section 4. Therefore we do bother about achieving "good constants".

3.2 Error bound for uniform OSπps

THEOREM 3.2 : Consider uniform OSπps(n ; $\underline{\lambda}$) from a population in which *i* is a specific unit. Then;

$$|\pi_i - \lambda_i| \leq \lambda_i \cdot \left(\vartheta_i + 3.1/n\right) \cdot \left(1 - \lambda/n\right)^{-1}, \quad \text{with} \tag{3.11}$$

$$\lambda = \max\{\lambda_1, \lambda_2, \ldots, \lambda_N\}, \tag{3.12}$$

$$\vartheta_i = 3 \cdot \frac{\log n}{\sqrt{n}} + \frac{\log(1/\lambda_i)}{\sqrt{n}} + 2 \cdot \left(\frac{\log(1/\lambda_i)}{\sqrt{n}}\right)^2, \tag{3.13}$$

provided that the following conditions are met;

$$n \cdot (1 + \vartheta_i + 1/n) \cdot (1 - \lambda \cdot (1 + \vartheta_i + 1/n)) \geq 1.25, \tag{3.14}$$

$$n \cdot (1 - \vartheta_i) \cdot (1 - \lambda \cdot (1 - \vartheta_i)) \geq 1.25, \tag{3.15}$$

$$1 + \vartheta_i + 1/n \leq 1/\lambda. \tag{3.16}$$

Proof: We follow the algorithm in Theorem 3.1. H and h are stated in (1.4), from where it is seen that (3.1) is satisfied. As stated in (1.5), $\theta_k = \lambda_k$, $k = 1, 2, \dots, N$, which together with (1.2) yields β_i in Step 1, $\beta_i = \log n + \log(1/\lambda_i)$. We chose α_i likewise;

$$\alpha_i = \log n + \log(1/\lambda_i). \tag{3.17}$$

By $(a+b)^2 \le 2\cdot(a^2+b^2)$, $\log n/\sqrt{n} < 1$ and (3.12) we get ;

$$\sqrt{n}\cdot\alpha_i + \alpha_i^2 \le n\cdot\vartheta_i. \tag{3.18}$$

In Step 2 we first note that (1.4) and (1.5) yield;

$$\sum_{k=1}^{N} H(\theta_k\cdot t) = t\cdot\sum_{k=1}^{N}\lambda_k = n\cdot t, \quad 0\le t\le 1/\lambda. \tag{3.19}$$

By this and (3.18), (3.5) becomes $\psi_i\cdot n \ge n + n\cdot\vartheta_i + 1$, which implies that (3.5) holds for;

$$\psi_i = 1 + \vartheta_i + 1/n. \tag{3.20}$$

In Step 3 is readily seen that δ in (3.6) equals n. Hence, an admissible ω_i is;

$$\omega_i = 1 - \vartheta_i. \tag{3.21}$$

In Step 4 we use the estimate;

$$\sum_{k=1}^{N} H(\theta_k\cdot t)\cdot(1 - H(\theta_k\cdot t)) = \sum_{k=1}^{N}\lambda_k\cdot t\cdot(1-\lambda_k\cdot t) \ge n\cdot t\cdot(1-\lambda\cdot t), \quad 0\le t\le 1/\lambda. \tag{3.22}$$

The function to the right in (3.22), $B(t) = n\cdot t\cdot(1-\lambda\cdot t)$, is convex (upwards). Thus its minimum over $[\omega_i, \psi_i]$ is attained in either of the end points. Hence, (3.8) is satisfied if $B(\omega_i)$ and $B(\psi_i)$ both exceed 1.25 which, by (3.20) and (3.21), leads to (3.14) and (3.15). We also used that $\psi_i \le 1/\lambda$, which is taken care of by (3.16).

In Step 5, when $\psi_i \le 1/\lambda$ the value of the sum in (3.9) can be calculated, which we chose as ρ_i;

$$\rho_i = \sum_{k=1}^{N}\lambda_k\cdot 1 - \{\max\{\lambda_1, \lambda_2, \dots, \lambda_N\} = n - \lambda. \tag{3.23}$$

Finally, insertion into (3.10) yields (3.11). ¤

3.3 Error bound for exponential OSπps

THEOREM 3.3: Consider exponential OSπps(n;$\underline{\lambda}$) from a population in which *i* is a specific unit, and let λ be according to (3.12). Then;

$$|\pi_i - \lambda_i| \le \lambda_i\cdot\frac{\vartheta_i + 1/n}{(1-\lambda)^2}\cdot\left(1 - \frac{\log[1/(1-\lambda)]}{n\cdot(1-\lambda)}\right)^{-1}, \quad \text{with} \tag{3.24}$$

$$\vartheta_i = 3\cdot\frac{\log(n/(1-\lambda))}{\sqrt{n}} + \frac{\log(1/\lambda_i)}{\sqrt{n}} + 2\cdot\left(\frac{\log(1/\lambda_i)}{\sqrt{n}}\right)^2, \tag{3.25}$$

provided that the following conditions are met;

$$n\cdot(1-\lambda+\vartheta_i+1/n)\cdot(1-\lambda) \ge 2.5. \tag{3.26}$$

$$n\cdot(1-\lambda-\vartheta_i)\cdot(1-\lambda) \ge 2.5. \tag{3.27}$$

$$1-\lambda \ge \vartheta_i + 1/n. \tag{3.28}$$

Proof: Again we follow the algorithm in Theorem 3.1. H and h are stated in (1.6), from where it is seen that (3.1) is satisfied. By (1.7) we have;

$$\theta_k = -\log(1-\lambda_k), \quad k = 1, 2, \dots, N. \tag{3.29}$$

The elementary inequalities below will be useful;

$$\lambda_k \le -\log(1-\lambda_k) \le \lambda_k/(1-\lambda_k) \le \lambda_k/(1-\lambda_k). \tag{3.30}$$

In Step 1 we start with the following estimate, where (3.30) is employed;

$$\sum_{k=1}^{N}\theta_k=\sum_{k=1}^{N}-\log(1-\lambda_k)\le\sum_{k=1}^{N}\frac{\lambda_k}{1-\lambda}=\frac{n}{1-\lambda}. \tag{3.31}$$

From (3.3), (3.31) and (3.29) + (3.30) follows that an admissible α_i is;

$$\alpha_i=\log(n/[(1-\lambda)\cdot\lambda_i]=\log(n/(1-\lambda))+\log(1/\lambda_i). \tag{3.32}$$

By using $(a+b)^2\le 2\cdot(a^2+b^2)$ in (3.32), and $\log[n/(1-\lambda)]<\sqrt{n}$ we get with ϑ_i as in (3.25);

$$\sqrt{n}\cdot\alpha_i+\alpha_i^2\le n\cdot\vartheta_i. \tag{3.33}$$

Next we prepare for Step 2. Since $H(t)=1-\exp(-t)$ is convex (upwards), so is

$$G(t)=\sum_{k=1}^{N}H(\theta_k\cdot t),\quad 0\le t<\infty. \tag{3.34}$$

Hence G(t) lies above its chords, which for the chord from t = 1 to t = 2 implies: $G(t)\ge G(1)+(t-1)\cdot[G(2)-G(1)]$, $1\le t\le 2$. In combination with $G(1)=n$ and

$$G(2)=\sum_{k=1}^{N}[1-(1-\lambda_k)^2]=\sum_{k=1}^{N}(2\cdot\lambda_k-\lambda_k^2)\ge n\cdot(2-\lambda), \tag{3.35}$$

we get $G(t)\ge n\cdot[1+(1-\lambda)\cdot(t-1)]$, $1\le t\le 2$, which by (3.33) yields that (3.5) is satisfied for;

$$\psi_i=1+\frac{\vartheta_i+1/n}{1-\lambda}\ge 1+\frac{\sqrt{n}\cdot\alpha_i+\alpha_i^2+1}{n\cdot(1-\lambda)}. \tag{3.36}$$

We prepare Step 3 by the following inequality, where (3.30) is used;

$$\delta=\sum_{k=1}^{N}\theta_k\cdot h(\theta_k)=\sum_{k=1}^{N}-\log(1-\lambda_k)\cdot e^{\log(1-\lambda_k)}\ge(1-\lambda)\cdot\sum_{k=1}^{N}\lambda_k=(1-\lambda)\cdot n. \tag{3.37}$$

From (3.37) and (3.33) is seen that an admissible ω_i is;

$$\omega_i=1-\vartheta_i/(1-\lambda). \tag{3.38}$$

We now turn to Step 4. By (1.6) and (1.7) we have;

$$\sum_{k=1}^{N}H(\theta_k\cdot t)\cdot(1-H(\theta_k\cdot t))=\sum_{k=1}^{N}(1-(1-\lambda_k)^t)\cdot(1-\lambda_k)^t\ge(1-\lambda)^t\cdot\sum_{k=1}^{N}(1-(1-\lambda_k)^t)\ge$$

The last sum is a convex (upwards) function of t and, hence, it lies above its chords. By using this for the chord from t=0 to t=2 we can continue the inequality as follows;

$$\ge(1-\lambda)^2\cdot\frac{t}{2}\cdot\sum_{k=1}^{N}(1-(1-\lambda_k)^2)=(1-\lambda)^2\cdot\frac{t}{2}\cdot\sum_{k=1}^{N}\lambda_k\cdot(2-\lambda_k)\ge\frac{t}{2}\cdot n\cdot(1-\lambda)^2\cdot \tag{3.39}$$

The final function (of t) in (3.39), $B(t)=0.5\cdot t\cdot n\cdot(1-\lambda)^2$, is linear. Thus its minimum over the interval $[\omega_i,\psi_i]$ is attained in either end point. Hence, (3.8) is satisfied if $B(\omega_i)$ and $B(\psi_i)$ both exceed 1.25, which leads to (3.26) and (3.27).

We turn to step 5 and exhibit a ρ_i which satisfies (3.9). Here we impose the condition $\psi_i\le 2$, which is taken care of by (3.28). Then we have by the estimates in (3.30);

$$\text{Right hand side in (3.9)}=\sum_{k=1}^{N}-\log(1-\lambda_k)\cdot(1-\lambda_k)^{\psi_i}+\log(1-\lambda)\ge$$

$$\ge(1-\lambda)^2\cdot\sum_{k=1}^{N}\lambda_k-\log[1/(1-\lambda)]=(1-\lambda)^2\cdot n-\log[1/(1-\lambda)]. \tag{3.40}$$

Hence, the right hand expression in (3.40) is an admissible ρ_i. Insertion of this ρ_i into (3.10) together with (3.29)+(3.30) and (3.33) yields the bound in (3.24). ¤

3.4 Error bound for Pareto OSπps

THEOREM 3.4: Consider Pareto OSπps(n;$\underline{\lambda}$) from a population in which *i* is a specific unit. Let λ and ϑ_i be according to (3.12) and (3.25). Then;

$$|\pi_i - \lambda_i| \le \lambda_i \cdot (\vartheta_i + 3.1/n) \cdot \frac{1.5}{(1-\lambda)^2} \cdot \left(1 - \frac{0.4}{n \cdot (1-\lambda)}\right)^{-1}, \tag{3.41}$$

provided that the following conditions are met;

$$n \cdot (1 - \lambda - \vartheta_i) \ge 1.8. \tag{3.42}$$

$$\vartheta_i + 1/n \le 0.17. \tag{3.43}$$

Proof: Again we follow the algorithm in Theorem 3.1. H and h are stated in (1.8), from where it is seen that (3.1) is satisfied. By (1.9);

$$\theta_k = \lambda_k/(1-\lambda_k), \quad k=1,2,\ldots,N. \tag{3.44}$$

In Step 1 we start with the estimate;

$$\sum_{k=1}^{N} \theta_k = \sum_{k=1}^{N} \frac{\lambda_k}{1-\lambda_k} \le \frac{1}{1-\lambda} \sum_{k=1}^{N} \lambda_k = \frac{n}{1-\lambda}. \tag{3.45}$$

From (3.45) and (3.3) is seen that;

$$\beta_i \le \log\left(\frac{n \cdot (1-\lambda_i)}{(1-\lambda) \cdot \lambda_i}\right) \le \log(n/(1-\lambda)) + \log(1/\lambda_i). \tag{3.46}$$

Hence, (3.32) yields an admissible α_i also here, and (3.33) applies. We turn to Step 2 and start with the estimate;

$$\sum_{k=1}^{N} H(\theta_k \cdot t) = t \cdot \sum_{k=1}^{N} \frac{\lambda_k}{1+\lambda_k \cdot (t-1)} \ge \frac{t \cdot n}{1+\lambda \cdot (t-1)} =$$

$$= n \cdot \left(1 + \frac{(1-\lambda)\cdot(t-1)}{1+\lambda \cdot (t-1)}\right) \ge n \cdot \left(1 + \frac{(1-\lambda)\cdot(t-1)}{t}\right), \quad t \ge 1. \tag{3.47}$$

From (3.47) and (3.33) is seen that (3.5) is satisfied if;

$$(1-\lambda) \cdot (\psi_i - 1)/\psi_i \ge \vartheta_i + 1/n \ge (\sqrt{n} \cdot \alpha_i + \alpha_i^2 + 1)/n. \tag{3.48}$$

Under the additional assumption $1 \le \psi_i \le 1.2$, (3.48) implies that (3.5) is satisfied for;

$$\psi_i = 1 + 1.2 \cdot (\vartheta_i + 1/n)/(1-\lambda). \tag{3.49}$$

In Step 3 we use the estimate;

$$\delta = \sum_{k=1}^{N} \theta_k \cdot h(\theta_k) = \sum_{k=1}^{N} \lambda_k \cdot (1-\lambda_k) \ge (1-\lambda) \cdot \sum_{k=1}^{N} \lambda_k = (1-\lambda) \cdot n. \tag{3.50}$$

From (3.50) is seen that the following ω_i makes (3.7) satisfied;

$$\omega_i = 1 - \vartheta_i/(1-\lambda). \tag{3.51}$$

Next we prepare Step 4. For $1 \le \psi_i \le 1.2$, we have;

$$\sum_{k=1}^{N} H(\theta_k \cdot t) \cdot (1 - H(\theta_k \cdot t)) = t \cdot \sum_{k=1}^{N} \frac{\lambda_k \cdot (1-\lambda_k)}{(1+\lambda_k \cdot (t-1))^2} \ge$$

$$\ge \frac{t \cdot (1-\lambda)}{(1+\lambda \cdot (t-1))^2} \cdot \sum_{k=1}^{N} \lambda_k = \frac{t \cdot n \cdot (1-\lambda)}{1.2^2} \ge 0.7 \cdot t \cdot n \cdot (1-\lambda), \quad 0 \le t \le 1.2. \tag{3.52}$$

The last function in (3.52), $B(t) = 0.7 \cdot t \cdot n \cdot (1-\lambda)$, is linear. Thus its minimum over $[\omega_i, \psi_i]$ is attained in either end point. Hence, (3.8) is satisfied if $B(\omega_i)$ and $B(\psi_i)$ both exceed 1.25, which is implied by (3.42). The assumption $1 \le \psi_i \le 1.2$ is taken care of by (3.43).

We turn to Step 5 and exhibit a ρ_i which makes (3.9) satisfied. Under $1 \le \psi_i \le 1.2$ we have;

Right hand side in (3.9) =

$$= \sum_{k=1}^{N} \frac{\lambda_k \cdot (1-\lambda_k)}{(1+\lambda_k \cdot (\psi_i - 1))^2} - \max_k \left\{ \frac{\lambda_k \cdot (1-\lambda_k)}{(1+\lambda_k \cdot (\psi_i - 1))^2} \right\} \ge 0.7 \cdot (1-\lambda) \cdot n - 0.25. \tag{3.53}$$

The last expression in (3.53) is an admissible ρ_i. Insertion into (3.10) of this ρ_i together with (3.44) and (3.33) yields the bound in (3.41). ¤

4 Asymptotic results

4.1 Generalities

The error bounds in Sections 2 and 3 yield information about the asymptotic behavior of OS π_i (n): s. The frame - work for limit consideration is as follows. A sequence, indexed by $q = 1,2,3,$ .., of $OS(n_q; \mathbf{F}^{(q)})$ from a population of size N_q is considered. A sub - or superscript q signifies that a quantity relates to the q:th situation. We aim at results of the following type.

$$\frac{\pi_i^{(q)}(n_q)}{F_i^{(q)}(\xi_i^{(q)})} \to 1, \quad \text{or equivalently} \quad \frac{F_i^{(q)}(\xi_i^{(q)})}{\pi_i^{(q)}(n_q)} \to 1, \quad \text{as } n_q \to \infty. \tag{4.1}$$

Results of type (4.1) for general OS may be derived along the following lines. Start from either of Theorems 2.1 and 2.2, and formulate conditions on N_q, $\mathbf{F}^{(q)}$ and n_q which imply that the theorem holds with vanishing approximation error as $q \to \infty$. However, the required conditions become quite involved when formulated for general OS situations. We therefore confine to the OSπps schemes of special interest.

4.2 Limit results for OSπps schemes

We consider OSπps with sample size n_q from a population of size N_q, in which i is a specific unit, $q = 1,2,3,\ldots$. The target inclusion probabilities and their maximal values are denoted;

$$\underline{\lambda}^{(q)} = (\lambda_1^{(q)}, \lambda_2^{(q)}, \ldots, \lambda_{N_q}^{(q)}) \quad \text{and} \quad \lambda^{(q)} = \max\{\lambda_1^{(q)}, \lambda_2^{(q)}, \ldots, \lambda_{N_q}^{(q)}\}, \quad q = 1,2,3,\ldots. \tag{4.2}$$

THEOREM 4.1: For uniform, exponential and Pareto OSπps holds;

$$\limsup_{q \to \infty} \left| \frac{\pi_i^{(q)}(n_q)}{\lambda_i^{(q)}} - 1 \right| \cdot \left(\frac{\log n_q + \log(1/\lambda_i^{(q)})}{\sqrt{n_q}} \right)^{-1} < \infty, \tag{4.3}$$

provided that the following conditions are met;

(i) $n_q \to \infty$, and hence $N_q \to \infty$, as $q \to \infty$, (4.4)

(ii) $\limsup_{q \to \infty} \lambda^{(q)} < 1$, (4.5)

(iii) $\lim_{q \to \infty} \log(1/\lambda_i^{(q)}) / \sqrt{n_q} = 0$. (4.6)

Proof: The claims in the theorem follow readily from Theorems 3.2, 3.3 and 3.4. The details are left to the reader. ¤

Remark 4.1: Condition (4.5) implies that sampling fractions stay away from 1 uniformly in q;

$$\limsup_{q \to \infty} n_q / N_q < 1. \tag{4.7}$$

This is realized from the inequality $\lambda^{(q)} \geq n_q/N_q$, which follows from (1.2). ¤

Remark 4.2: If target inclusion probabilities are of the same order for all population units (uniformly in q), i.e. if the following condition is met;

$$\lambda^{(q)} \leq c \cdot \min\{\lambda_1^{(q)}, \lambda_2^{(q)}, \ldots, \lambda_{N_q}^{(q)}\}, \text{ for some } c < \infty, \quad q = 1,2,3,\ldots, \tag{4.8}$$

we have for any i;

$$\limsup_{q\to\infty} \left| \frac{\pi_i^{(q)}(n_q)}{\lambda_i^{(q)}} - 1 \right| \cdot \left(\frac{\log N_q}{\sqrt{n_q}} \right)^{-1} < \infty, \quad i = 1,2,\ldots,N_q, \tag{4.9}$$

provided that (4.4), (4.5) and $\limsup_{q\to\infty} (\log N_q)/n_q = 0$ hold. If also sampling rates stay away from 0 uniformly in q, i.e. $n_q/N_q \geq d > 0$, $q = 1,2,3,\ldots$, we have for any i;

$$\limsup_{q\to\infty} \left| \frac{\pi_i^{(q)}(n_q)}{\lambda_i^{(q)}} - 1 \right| \cdot \left(\frac{\log n_q}{\sqrt{n_q}} \right)^{-1} < \infty, \quad i = 1,2,\ldots,N_q. \tag{4.10}$$

The result (4.9) is realized as follows. By (4.8), $\lambda_i \geq \lambda^{(q)}/c$. Combined with $\lambda^{(q)} \geq n_q/N_q$ this yields $\lambda_i \geq n_q/(c \cdot N_q)$, which in turn yields $\log(1/\lambda_i) \leq \log N_q + \log c$, which implies (4.9). ¤

4.3 On estimator bias under OSπps sampling

We consider estimation of the population total $\tau(\mathbf{y}) = y_1 + y_2 + \ldots + y_N$ from $\mathbf{y}$-observations of the units in an OSπps sample with target inclusion probabilities $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$. Let $I_1, I_2, \ldots, I_N$ denote the sample inclusion indicators, i.e. $I_k = 1$ if unit k is sampled, and 0 otherwise. The natural estimator of $\tau(\mathbf{y})$, and the one advised in Rosén (1977b), is;

$$\hat{\tau}(\mathbf{y}) = \sum_{k \in \text{sample}} \frac{y_k}{\lambda_k} = \sum_{k=1}^{N} \frac{y_k}{\lambda_k} \cdot I_k. \tag{4.11}$$

By taking expectation in (4.11) we get;

$$E[\hat{\tau}(\mathbf{y})] = \sum_{k=1}^{N} y_k \cdot \frac{\pi_k}{\lambda_k} = \tau(\mathbf{y}) + \sum_{k=1}^{N} y_k \cdot \left(\frac{\pi_k}{\lambda_k} - 1 \right). \tag{4.12}$$

Remark 4.2 tells that, under general conditions, $(\pi_k/\lambda_k - 1)$ is of order at most $O(\log n/\sqrt{n})$. This together with (4.12) yields that under general conditions we have;

$$E[\hat{\tau}(\mathbf{y})] = \tau(\mathbf{y}) \cdot [1 + O(\log n/\sqrt{n})]. \tag{4.13}$$

(4.13) shows that the estimator (4.11) is consistent under general conditions. It should be noted that the error order estimate in (4.13) is conservative, i.a. to the effect that no regard is paid to cancellation effects from alternating signs of $(\pi_k/\lambda_k - 1)$. Moreover, as discussed in next sub-section, the error order $O(\log n/\sqrt{n})$ for $(\pi_k/\lambda_k - 1)$ is probably also conservative. In any case, (4.13) shows that (4.11) yields consistent estimation under very general conditions.

4.4 Comments on the order of the approximation error

A natural question is if the error order $O(\log n/\sqrt{n})$ for $(\pi_k/\lambda_k - 1)$ is the correct one. Our conjecture is, regrettably, that it is *not*. We believe, based on arguments provided by colleagues, that the correct error order rather is $O(1/n)$. However, as indicated below the approach in this paper cannot yield better than $O(\log n/\sqrt{n})$.

The estimates in (2.11) and (2.12) disregard the event $\omega_i < Q_i < \psi_i$. As a consequence, the error bound (2.6) is at least $P(\omega_i < Q_i < \psi_i)$. For this term to be small, ω_i and ψ_i shall lie close. This is counterbalanced by the desire that $P(A_i < n)$ and $P(B_i \geq n)$ in (2.11) and (2.12) should be close to 1. We believe that the balancing is carried out fairly optimally when yielding order $O(\log n/\sqrt{n})$. Hence, to obtain a sharper bound, one must be more deep-going in the analysis of what happens near ξ. This, however, increases the problem difficulty dramatically.

Other indications that the correct error order is not reached are as follows. (i) Simulation findings in Rosén (1997b) on estimator bias comply better with $O(1/n)$. (ii) By a "direct" analysis, Ohlsson (1990) shows that for uniform OSπps, (4.13) holds with error order $O(1/\sqrt{n})$.

5 On exact formulas for OSFS inclusion probabilities

In this section we derive some exact formulas for OSFS inclusion probabilities. One aim is to illustrate the following point.

Although OSFS is simple in definition and implementation, computation of inclusion probabilities is exceedingly hard if the sample size is not very small. (5.1)

However, for a certain non-trivial situation, computationally manageable formulas can be derived, and this is a second aim. The resulting formulas are then used in a numerical study of the goodness of the approximation (1.13).

5.1 Inclusion probabilities for general OS schemes

A suggestive language for describing an OS(n; F) sample is as follows. The population units have independent random life lengths Q_i with distributions F_i, $i = 1, 2, \dots, N$. The sample consists of the n units which die first. Set, where we presume that the arguments in δ are different;

$$\delta(\nu; i_1, i_2, \dots, i_\nu) = P(\text{unit } i_1 \text{ dies first, } i_2 \text{ dies second}, \dots, i_\nu \text{ is the } \nu\text{:th unit to die}). \quad (5.2)$$

We have;

$$\pi_i(n) = \sum_{\nu=1}^{n} \sum_{i_1, i_2, \dots, i_{\nu-1}} \delta(\nu; i_1, i_2, \dots, i_{\nu-1}, i), \quad i = 1, 2, \dots, N. \quad (5.3)$$

An expression for the δ:s is stated in (5.5) below, where we use the notation;

$$\bar{F}(t) = 1 - F(t), \quad 0 \le t < \infty. \quad (5.4)$$

(5.5) is justified as follows. Death order $i_1, i_2, \dots, i_\nu$ occurs if, for $t_1 < t_2 < \dots < t_\nu$ and infinitesimal $dt_1, dt_2, \dots, dt_\nu$, unit i_1 dies during the time interval $[t_1, t_1 + dt_1)$, unit i_2 during $[t_2, t_2 + dt_2)$, ..., unit i_ν during $[t_\nu, t_\nu + dt_\nu)$, while the other units survive time $t_\nu + dt_\nu$. Since life lengths are independent we get by summing (= integrating) over the possibilities for $(t_1, t_2, \dots, t_\nu)$;

$$\delta(\nu; i_1, i_2, \dots, i_\nu) = \int_0^\infty f_{i_1}(t_1)dt_1 \int_{t_1}^\infty f_{i_2}(t_2)dt_2 \dots \int_{t_{\nu-1}}^\infty f_{i_\nu}(t_\nu) \cdot \prod_{k \ne i_1, i_2, \dots, i_\nu} \bar{F}_k(t_\nu)\, dt_\nu . \quad (5.5)$$

5.2 Inclusion probabilities for OSFS schemes

By (1.1), (5.5) takes the following form for OSFS(n; H; $\underline{\theta}$);

$$\delta(\nu; i_1, i_2, \dots, i_\nu) = \int_0^\infty \theta_{i_1} h(\theta_{i_1} \cdot t_1)dt_1 \int_{t_1}^\infty \theta_{i_2} h(\theta_{i_2} \cdot t_2) \dots \int_{t_{\nu-1}}^\infty \theta_{i_\nu} h(\theta_{i_\nu} \cdot t_\nu) \cdot \prod_{k \ne i_1, i_2, \dots, i_\nu} \bar{H}(\theta_i \cdot t_\nu)\, dt_\nu . \quad (5.6)$$

Formulas (5.3) + (5.5) provide one way to compute $\pi_i(n)$. An alternative is presented below. Introduce the events, where i is a fixed unit and dt is infinitesimal;

$$C(t; dt) = \text{unit } i \text{ dies during the time interval } [t, t+dt), \quad 0 \le t < \infty, \quad (5.7)$$

$$D(\nu; t) = \nu \text{ of units } \{1, 2, \dots, N\} \setminus \{i\} \text{ die during } [0, t), \quad \nu = 0, 1, \dots, N-1, \quad 0 \le t < \infty. \quad (5.8)$$

The following holds: (i) $D(\nu;t)$ and $D(\mu;t)$ are disjoint for $\nu\neq\mu$. (ii) $C(t;dt)$ and $C(s;dt)$ are disjoint for $t\neq s$. (iii) $C(t;dt)$ and $D(\nu;t)$ are independent. (iv) Unit *i* is sampled if $C(t;dt)$ and $D(\nu;t)$ occur with $\nu < n$. Combination of (i)-(iv) yields;

$$\pi_i(n)=\sum_{\nu=0}^{n-1}\int_0^{\infty}P[D(\nu;t)]\cdot\theta_i\cdot h(\theta_i\cdot t)\,dt\ . \tag{5.9}$$

We pursue this approach in Section 5.2.3 under special assumptions on the intensities. First we consider application of (5.3) to a case with general intensities.

5.2.1 Inclusion probabilities for exponential OSFS

As specified in (1.6), for exponential OSπps $\overline{H}(t)=h(t)=e^{-t}$, $0\leq t<\infty$. Then the integral (5.6) is straightforward to evaluate, yielding (5.10) below. Alternatively (5.10) can be derived by well-known results for Poisson processes. See e.g. Rosén (1997a), Remark 2.2.

$$\delta(\nu;i_1,i_2,\ldots,i_\nu)=\frac{\theta_{i_1}}{\sum_{k=1}^{N}\theta_k}\cdot\frac{\theta_{i_2}}{\sum_{k\neq i_1}\theta_k}\cdot\ \ldots\ \cdot\frac{\theta_{i_{\nu-1}}}{\sum_{k\neq i_1,i_2,\ldots,i_{\nu-2}}\theta_k}\cdot\frac{\theta_{i_\nu}}{\sum_{k\neq i_1,i_2,\ldots,i_{\nu-1}}\theta_k}\ . \tag{5.10}$$

Even if (5.10) is simple, the associated formula for $\pi_i(n)$ obtained by combining (5.3) and (5.10) is computationally unfeasible already for quite small n, which illustrates (5.1).

5.2.2 Inclusion probabilities for Pareto OSFS with n = 1

For Pareto OSFS matters become complicated already for n = 1, and we confine to that case. As stated in (1.8), $\overline{H}(t)=1/(1+t)$ and $h(t)=1/(1+t)^2$, $0\leq t<\infty$. Hence, (5.6) and (5.3) yield;

$$\pi_i(1)=\delta(1;i)=\int_0^{\infty}\frac{\theta_i\,dt}{(1+\theta_i\cdot t)^2\cdot\prod_{k\neq i}(1+\theta_k\cdot t)}=\int_0^{\infty}\frac{dt}{(1+t)^2\cdot\prod_{k\neq i}(1+t\cdot\theta_k/\theta_i)}\ . \tag{5.11}$$

The integral (5.11) is treated in Section 5.2.4. Formulas (5.11), (5.21) and (5.22) yield, when ***all intensities are different***;

$$\pi_i(1)=\frac{\theta_i^{N-1}}{\Omega(i)}+\theta_i\cdot\sum_{k\neq i}\frac{\theta_k^{N-1}\cdot\log\theta_k/\theta_i}{(\theta_k-\theta_i)\cdot\Omega(k)},\quad\text{where}\quad\Omega(k)=\prod_{j\neq k}(\theta_k-\theta_j)\ . \tag{5.12}$$

(5.12) certainly exemplifies the point in (5.1).

5.2.3 Inclusion probabilities for OSFS with one odd unit

We consider the following particular situation, called a ***situation with one odd unit***: All population units but one have the same intensity. Let 1 be the "odd" unit, with intensity θ_1, while units $2,\ldots,N$ have intensities θ_2. Then, from (5.8) with $i=1$ we get;

$$P[D(\nu;t)]=\binom{N-1}{\nu}\cdot[1-\overline{H}(\theta_2\cdot t)]^{\nu}\cdot\overline{H}(\theta_2\cdot t))^{N-1-\nu}=$$

$$=\binom{N-1}{\nu}\cdot\sum_{\rho=0}^{\nu}\binom{\nu}{\rho}\cdot(-1)^{\rho}\cdot\overline{H}(\theta_2\cdot t)^{N-1-\nu+\rho}\ . \tag{5.13}$$

Combination of (5.9) and (5.13) yields;

$$\pi_1(n)=\sum_{\nu=0}^{n-1}\binom{N-1}{\nu}\cdot\sum_{\rho=0}^{\nu}\binom{\nu}{\rho}\cdot(-1)^{\rho}\cdot\Psi(N-1-\nu+\rho;\kappa),\quad\text{where} \tag{5.14}$$

$$\kappa=\theta_2/\theta_1\ , \tag{5.15}$$

$$\Psi(m;\kappa) = \int_0^\infty \overline{H}(\theta_2 \cdot t)^m \cdot \theta_1 \cdot h(\theta_1 \cdot t)\,dt = \int_0^\infty \overline{H}(\kappa \cdot t)^m \cdot h(t)\,dt. \tag{5.16}$$

In (5.18)-(5.20) we present formulas for $\Psi(m;\kappa)$ for the OSFS schemes of particular interest.

Once $\pi_1(n)$ is derived, the other inclusion probabilities are readily deduced by using the following facts: (i) $\pi_2(n) = \ldots = \pi_N(n)$, (ii) inclusion probabilities sum to the sample size;

$$\pi_i(n) = \frac{n - \pi_1}{n-1}, \; i = 2,3,\ldots,N. \tag{5.17}$$

Ψ for uniform OSFS

Here we have $\overline{H}(t) = 1 - \min(t,1)$ and $h(t) = \mathbf{1}_{[0,1]}(t)$, $0 \le t < \infty$, which yields;

$$\Psi(m;\kappa) = \int_0^1 [1 - \min(\kappa \cdot t, 1)]^m\,dt = \frac{1 - [1 - \min(\kappa,1)]^{m+1}}{\kappa \cdot (m+1)}. \tag{5.18}$$

Ψ for exponential OSFS

Here we have $\overline{H}(t) = h(t) = e^{-t}$, $0 \le t < \infty$, which yields;

$$\Psi(m;\kappa) = \int_0^\infty e^{-\kappa \cdot m \cdot t} \cdot e^{-t}\,dt = \frac{1}{\kappa \cdot m + 1}. \tag{5.19}$$

Ψ for Pareto OSFS

Here we have $\overline{H}(t) = 1/(1+t)$ and $h(t) = 1/(t+1)^2$, $0 \le t < \infty$, which yields;

$$\Psi(m;\kappa) = \int_0^\infty \frac{dt}{(1+\kappa \cdot t)^m \cdot (1+t)^2}\,dt. \tag{5.20}$$

The integral (5.20) is, in a more general version, considered in next section. Formulas (5.23) and (5.24) together with the relation $\Psi(m;\kappa) = I(m;\underline{\kappa})$ provide expressions for $\Psi(m;\kappa)$.

5.2.4 Evaluation of an integral

Here we consider the integral.

$$I(m,\underline{\kappa}) = \int_0^\infty \frac{dt}{(1+t)^2 \cdot \prod_{k=1}^{m} (1 + \kappa_k \cdot t)}. \tag{5.21}$$

Explicit expressions for $I(m;\underline{\kappa})$ depend on the number of κ: s that agree. The results for two particular cases are stated below. Empty product yields 1 and empty summation 0.

Case 1: $\kappa_1, \kappa_2, \ldots, \kappa_m$ are mutually different as well as different from 1.

$$I(m;\underline{\kappa}) = \left[\prod_{k=1}^{m} (1 - \kappa_k)\right]^{-1} + \sum_{k=1}^{m} \frac{\kappa_k^m \cdot \log \kappa_k}{(1-\kappa_k)^2} \cdot \left[\prod_{\ell \ne k} (\kappa_k - \kappa_\ell)\right]^{-1}. \tag{5.22}$$

Case 2: All $\kappa_1, \kappa_2, \ldots, \kappa_m$ have the same value κ, which differs from 1.

$$I(m;\underline{\kappa}) = \frac{1}{1-\kappa} + \frac{m \cdot \kappa}{(1-\kappa)^{m+1}} \cdot \left(\log \kappa + \sum_{k=1}^{m-1} \frac{(1-\kappa)^k}{k}\right), \quad 0 \le \kappa < \infty, \quad \kappa \ne 1. \tag{5.23}$$

Power series expansion, in powers of $1-\kappa$, in (5.23) leads to the following alternative formula;

$$I(m;\underline{\kappa}) = \frac{1}{m+1} + m \cdot \sum_{k=1}^{\infty} \frac{(1-\kappa)^k}{(m+k) \cdot (m+k+1)}, \quad 0 < \kappa < 2. \tag{5.24}$$

Proof: $I(m;\underline{\kappa})$ has rational integrand, and can thus be evaluated by partial fraction expansion. In Case 1 the expansion has the following structure;

$$\left[(1+t)^2\cdot\prod_{k=1}^{m}(1+\kappa_k\cdot t)\right]^{-1}=\frac{A}{(1+t)^2}+\frac{B}{1+t}+\sum_{k=1}^{m}\frac{C_k}{1+\kappa_k\cdot t}. \tag{5.25}$$

The values for A and C_k, which are stated in (5.26), are obtained by the "standard method". To obtain the value for B, multiply (5.25) by (1+t), and let t tend to ∞. Then the left hand side and, hence, the right hand side tends to 0, which yields B.

$$A=\left[\prod_{k=1}^{m}(1-\kappa_k)\right]^{-1},\quad C_k=\left[(1-\frac{1}{\kappa_k})^2\cdot\prod_{\ell\neq k}(1-\frac{\kappa_\ell}{\kappa_k})\right]^{-1},\ k=1,\ldots,m,\quad B=-\sum_{k=1}^{m}\frac{C_k}{\kappa_k}. \tag{5.26}$$

By integrating (5.25);

$$I(m;\underline{\kappa})=A\cdot\int_0^\infty\frac{dt}{(1+t)^2}+\sum_{k=1}^{m}C_k\cdot\int_0^\infty\left(\frac{1}{1+\kappa_k\cdot t}-\frac{1}{\kappa_k\cdot(1+t)}\right)dt=A+\sum_{k=1}^{m}\frac{C_k\cdot\log\kappa_k}{\kappa_k}. \tag{5.27}$$

(5.27) and (5.26) now yield (5.22). In Case 2 the partial fraction expansion has the structure;

$$\left[(1+t)^2\cdot(1+\kappa\cdot t)^m\right]^{-1}=\frac{A}{(1+t)^2}+\frac{B}{1+t}+\sum_{k=1}^{m}\frac{C_k}{(1+\kappa\cdot t)^k}. \tag{5.28}$$

The values for A, B and C_k are as follows, where A is obtained by the standard method;

$$A=(1-\kappa)^{-m},\quad B=-\frac{m\cdot\kappa}{(1-\kappa)^{m+1}},\quad C_k=\frac{\kappa^2\cdot(m-k+1)}{(1-\kappa)^{m-k+2}},\ k=1,2,\ldots,m. \tag{5.29}$$

Straightforward algebra yields;

$$[(1+t)^2\cdot(1+\kappa\cdot t)^m]^{-1}-\frac{(1-\kappa)^{-m}}{(1+t)^2}=-\frac{1}{1+t}\cdot\frac{\kappa}{(1-\kappa)^{m+1}}\cdot\sum_{k=1}^{m}\left(\frac{1-\kappa}{1+\kappa\cdot t}\right)^k. \tag{5.30}$$

By combining (5.28) and (5.30) and letting t tend to -1, the value of B is obtained. Again by straightforward algebra, using (5.30);

$$\left[(1+t)^2\cdot(1+\kappa\cdot t)^m\right]^{-1}-\frac{(1-\kappa)^{-m}}{(1+t)^2}+\frac{1}{1+t}\cdot\frac{m\cdot\kappa}{(1-\kappa)^{m+1}}=$$

$$=\frac{\kappa^2}{(1-\kappa)^{m+2}}\cdot\sum_{k=1}^{m}\sum_{\ell=1}^{k}\left(\frac{1-\kappa}{1+\kappa\cdot t}\right)^\ell=\frac{\kappa^2}{(1-\kappa)^{m+2}}\cdot\sum_{k=1}^{m}(m-k+1)\cdot\left(\frac{1-\kappa}{1+\kappa\cdot t}\right)^k. \tag{5.31}$$

Now (5.31) yields C_k. Note that $B=-C_1/\alpha$. Hence we have the following expansion;

$$\left[(1+t)^2\cdot(1+\kappa\cdot t)^m\right]^{-1}=\frac{A}{(1+t)^2}+C_1\cdot\left(\frac{1}{1+\kappa\cdot t}-\frac{1}{\kappa\cdot(1+t)}\right)+\sum_{k=2}^{m}\frac{C_k}{(1+\kappa\cdot t)^k}. \tag{5.32}$$

By integrating (5.32);

$$I(m;\underline{\kappa})=A\cdot\int_0^\infty\frac{dt}{(1+t)^2}+C_1\cdot\int_0^\infty\left(\frac{1}{1+\kappa\cdot t}-\frac{1}{\kappa\cdot(1+t)}\right)dt+\sum_{k=2}^{m}C_k\cdot\int_0^\infty\frac{dt}{(1+\kappa\cdot t)^k}=$$

$$=A+\frac{C_1\cdot\log\kappa}{\kappa}+\frac{1}{\kappa}\cdot\sum_{k=1}^{m-1}\frac{C_{k+1}}{k}. \tag{5.33}$$

Combination of (5.33) and (5.29) yields (5.23). ¤

5.3 Numerical findings for OSπps

5.3.1 Introduction and conclusions

Numerical studies of OSπps inclusion probabilities may yield insight into the approximation goodness in (1.13), especially one may hope for answers to the following questions.

How large must the sample size be in a practical OSπps situation in order that the true inclusion probabilities should be "sufficiently" close to the target ones? (5.34)

What rate of convergence in (1.15) is indicated by numerical findings? (5.35)

For such a study one would like to dispose of numerically manageable formulas for a great variety of OSπps situations. However, we master just one nontrivial situation, the one in Section 5.2.3 for OSπps with one odd unit. Our study is therefore confined to this type of situation. Accordingly we cannot draw comprehensive conclusions from the numerical findings. Nevertheless we mean that they indicate quite strongly that answers to question (5.34) are given by (5.36) and (5.37) below. These are our main conclusions from the numerical study. The reader may agree or disagree after having looked at the subsequent tables.

For Pareto OSπps, $\pi_i(n)$ differs only negligibly from λ_i if $\min(n, N-n) \geq 5$. (5.36)

Inclusion probabilities for uniform and exponential OSπps also converge rapidly to target values, but not as fast as for Pareto OSπps. (5.37)

Answers to question (5.35) are, however, out of reach for at least the following reason. Even if the formulas in Section 5.2.3 hold mathematically for arbitrarily large N and n, the binomial coefficients make them numerically unstable already for fairly small n and N.

5.3.2 Computation procedure for OSπps with one odd unit

Generally, approximation goodness in (1.13) depends on the values of the parameters $\underline{\lambda}$, N, and n in the situation under consideration. As regards λ-values we make the following observations. Condition (4.5) indicates that the closer the maximal λ_i lies to 1, the larger n-value is needed to achieve good approximation. On the other hand, in practice, λ-values very close to 1 are avoided by forming a "certainty stratum" for units with very large size measures. Moreover, the $\log(1/\lambda_i)$ term in the error bound in (4.3) indicates that large n-values are needed to hold down the relative approximation error for a small λ_i. Therefore, when varying n and N we think it is instructive to consider situations with (i) prescribed maximal λ, (ii) prescribed ratio between the maximal and minimal λ. In terms of size measures the latter means prescribed ratio between the largest and smallest size measure values. The bigger the prescribed values in (i) and (ii) are, the larger n is presumably required for good approximation.

For OSπps with one odd unit there are two different λ-values, λ_1 for the odd unit and λ_2 for the non-odd units. By (1.2), λ_1 and λ_2 relate as follows;

$$\lambda_2 = (n-\lambda_1)/(N-1). \qquad (5.38)$$

The computation procedure runs as follows. When one of λ_1 and λ_2 is decided on, the other follows from (5.38). Having λ_1 and λ_2, the associated intensities $\theta_1 = H^{-1}(\lambda_1)$ and $\theta_2 = H^{-1}(\lambda_2)$ can be computed, and they yield κ in (5.15). Then the formulas in Section 5.2.3 can be set in work to compute $\pi_1(n)$. Once $\pi_1(n)$ is known, $\pi_2(n)$ for the non-odd units is obtained by using: (i) the non-odd π:s are equal, (ii) inclusion probabilities sum up to the sample size;

$$\pi_2(n) = (n - \pi_1(n))/(N-1). \qquad (5.39)$$

As regards absolute and relative approximation errors, note that (5.38) and (5.39) yield;

$$\pi_2(n) - \lambda_2 = -(\pi_1(n) - \lambda_1)/(N-1)\,, \tag{5.40}$$

$$\frac{\pi_2(n)}{\lambda_2} - 1 = \frac{\lambda_1}{\lambda_2 \cdot (N-1)} \cdot \left(\frac{\pi_1(n)}{\lambda_1} - 1 \right). \tag{5.41}$$

These formulas show that the inclusion probability for the odd unit has the lager absolute error and, unless λ_2 is very small, also the larger relative error.

The numerical study comprises two blocks, "situations with fixed λ_1" respectively "situations with fixed ratio λ_1 / λ_2". Under these fixations, N and n were varied, and the corresponding inclusion probabilities were computed.

Situations with fixed λ_1

For prescribed λ_1 - values, different population sizes N were considered and for each N the sample size was run over all possible values, $n = 1, 2, \ldots, N-1$. Note the following. For the considered λ_1 - values, 0.2, 0.5 and 0.7, $\lambda_1 \geq \lambda_2$ for $N \geq 5$. Hence, λ_1 is the largest target inclusion probability.

Tables 1 - 5 present $\pi_1(n)$ - values for uniform, exponential and Pareto OSπps, denoted by $\pi_1(n;U)$, $\pi_1(n;E)$ and $\pi_1(n;P)$ respectively. Values for λ_1/λ_2, are also presented. We could not go further than $N=30$, because numerical instability turned up for larger N - values.

Situations with fixed λ - ratio

Here values for the ratio $\rho = \lambda_1/\lambda_2$ were prescribed. By (1.2) we have;

$$\lambda_1 = (\rho \cdot n)/(N-1+\rho) \quad \text{and} \quad \lambda_2 = n/(N-1+\rho)\,. \tag{5.42}$$

Note that a given $\rho = \lambda_1 / \lambda_2$ is not compatible with arbitrarily large sample sizes, since (5.42) may lead to λ_1 or λ_2 that exceed 1. "Misses" are reported by blank cells in the tables.

For given ρ, the values of λ_1 and $\pi_1(n)$ were computed for a variety of N - and n - values, for uniform, exponential and Pareto OSπps. From Tables 1-5 is seen that approximation errors (as can be expected) are largest at the ends of the region of possible sample sizes, i.e. for n: s that are small or close to N. Since sampling rates very close to 1 are quite uninteresting from a sampling practical point of view, we confined to sample sizes up to 10, in order to mitigate numerical instability problems. We presume that approximations are only better as n increases, until it comes very close to N.

To test the approximation goodness in as demanding situations as possible, we used as large as possible ρ - choices, under the restriction that sample sizes up to $n = 5$ should be admissible. The results are presented in Tables 6 - 11. For $N = 100$, the blank cells for n = 7 and 8 depend on numerical instability.

5.3.3 Results for situations with fixed λ_1

Table 1.	**Inclusion probabilities for some different λ_1**								**N=5**			
n	**λ_1 = 0.2**				**λ_1 = 0.5**				**λ_1 = 0.7**			
	λ_1/λ_2	**$\pi_1(n;U)$**	**$\pi_1(n;E)$**	**$\pi_1(n;P)$**	**λ_1/λ_2**	**$\pi_1(n;U)$**	**$\pi_1(n;E)$**	**$\pi_1(n;P)$**	**λ_1/λ_2**	**$\pi_1(n;U)$**	**$\pi_1(n;E)$**	**$\pi_1(n;P)$**
1	1.0	0.200	0.200	0.200	4.0	0.610	0.565	0.535	9.3	0.807	0.794	0.766
2	0.4	0.178	0.187	0.196	1.3	0.529	0.510	0.505	2.2	0.741	0.720	0.712
3	0.3	0.171	0.176	0.189	0.8	0.480	0.492	0.495	1.2	0.719	0.704	0.703
4	0.2	0.168	0.140	0.129	0.6	0.457	0.466	0.465	0.8	0.679	0.695	0.693

Table 2. Inclusion probabilities for some different λ_1 N=10

n	$\lambda_1 = 0.2$				$\lambda_1 = 0.5$				$\lambda_1 = 0.7$			
	λ_1/λ_2	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1/λ_2	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1/λ_2	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$
1	2.3	0.224	0.210	0.203	9.0	0.623	0.574	0.537	21.0	0.811	0.798	0.766
2	1.0	0.200	0.200	0.200	3.0	0.564	0.523	0.508	4.8	0.748	0.726	0.712
3	0.6	0.193	0.196	0.199	1.8	0.535	0.510	0.503	2.7	0.736	0.712	0.705
4	0.5	0.189	0.194	0.199	1.3	0.514	0.504	0.501	1.9	0.729	0.706	0.703
5	0.4	0.187	0.192	0.199	1.0	0.500	0.500	0.500	1.5	0.721	0.703	0.702
6	0.3	0.186	0.190	0.198	0.8	0.491	0.497	0.499	1.2	0.711	0.701	0.701
7	0.3	0.185	0.187	0.196	0.7	0.485	0.493	0.497	1.0	0.700	0.700	0.700
8	0.2	0.185	0.181	0.189	0.6	0.480	0.489	0.492	0.9	0.690	0.699	0.698
9	0.2	0.184	0.151	0.129	0.5	0.476	0.472	0.463	0.8	0.683	0.697	0.690

Table 3. Inclusion probabilities for some different λ_1 N=15

n	$\lambda_1 = 0.2$				$\lambda_1 = 0.5$				$\lambda_1 = 0.7$			
	λ_1/λ_2	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1/λ_2	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1/λ_2	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$
1	3.5	0.232	0.213	0.204	14.0	0.626	0.577	0.538	32.7	0.812	0.799	0.766
2	1.6	0.207	0.203	0.201	4.7	0.571	0.527	0.508	7.5	0.750	0.728	0.712
3	1.0	0.200	0.200	0.200	2.8	0.547	0.514	0.503	4.3	0.739	0.714	0.705
4	0.7	0.196	0.198	0.200	2.0	0.530	0.508	0.502	3.0	0.733	0.708	0.703
5	0.6	0.194	0.197	0.200	1.6	0.518	0.505	0.501	2.3	0.729	0.705	0.702
6	0.5	0.193	0.196	0.200	1.3	0.509	0.503	0.500	1.8	0.724	0.704	0.701
7	0.4	0.192	0.195	0.199	1.1	0.503	0.501	0.500	1.6	0.719	0.702	0.701
8	0.4	0.191	0.195	0.199	0.9	0.498	0.499	0.500	1.3	0.714	0.701	0.701
9	0.3	0.191	0.194	0.199	0.8	0.494	0.498	0.500	1.2	0.708	0.701	0.700
10	0.3	0.190	0.193	0.199	0.7	0.491	0.497	0.499	1.1	0.703	0.700	0.700
11	0.3	0.190	0.191	0.198	0.7	0.489	0.495	0.498	1.0	0.698	0.700	0.700
12	0.2	0.190	0.189	0.196	0.6	0.487	0.493	0.497	0.9	0.694	0.700	0.699
13	0.2	0.190	0.183	0.190	0.6	0.485	0.490	0.492	0.8	0.691	0.700	0.698
14	0.2	0.189	0.155	0.130	0.5	0.484	0.476	0.462	0.7	0.688	0.699	0.689

Table 4. Inclusion probabilities for some different λ_1 N=20

n	$\lambda_1 = 0.2$				$\lambda_1 = 0.5$				$\lambda_1 = 0.7$			
	λ_1/λ_2	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1/λ_2	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1/λ_2	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$
1	4.8	0.235	0.214	0.204	19.0	0.628	0.578	0.538	44.3	0.812	0.799	0.766
2	2.1	0.211	0.205	0.201	6.3	0.575	0.528	0.508	10.2	0.751	0.728	0.712
3	1.4	0.204	0.202	0.200	3.8	0.552	0.516	0.504	5.8	0.740	0.715	0.705
4	1.0	0.200	0.200	0.200	2.7	0.537	0.510	0.502	4.0	0.735	0.709	0.703
5	0.8	0.198	0.199	0.200	2.1	0.526	0.507	0.501	3.1	0.731	0.706	0.702
6	0.7	0.197	0.198	0.200	1.7	0.518	0.505	0.501	2.5	0.728	0.705	0.701
7	0.6	0.196	0.198	0.200	1.5	0.511	0.503	0.500	2.1	0.724	0.703	0.701
8	0.5	0.195	0.197	0.200	1.3	0.507	0.502	0.500	1.8	0.721	0.702	0.701
9	0.4	0.194	0.197	0.200	1.1	0.503	0.501	0.500	1.6	0.717	0.702	0.700
10	0.4	0.194	0.196	0.200	1.0	0.500	0.500	0.500	1.4	0.713	0.701	0.700
11	0.4	0.194	0.196	0.200	0.9	0.498	0.499	0.500	1.3	0.710	0.701	0.700
12	0.3	0.193	0.195	0.199	0.8	0.496	0.499	0.500	1.2	0.706	0.700	0.700
13	0.3	0.193	0.195	0.199	0.8	0.494	0.498	0.500	1.1	0.703	0.700	0.700
14	0.3	0.193	0.194	0.199	0.7	0.493	0.497	0.499	1.0	0.700	0.700	0.700
15	0.3	0.193	0.193	0.199	0.7	0.491	0.496	0.499	0.9	0.698	0.700	0.700
16	0.2	0.192	0.192	0.198	0.6	0.490	0.495	0.498	0.9	0.695	0.700	0.700
17	0.2	0.192	0.190	0.196	0.6	0.489	0.494	0.496	0.8	0.694	0.700	0.699
18	0.2	0.192	0.184	0.190	0.5	0.489	0.491	0.492	0.8	0.692	0.700	0.697
19	0.2	0.192	0.158	0.130	0.5	0.488	0.478	0.462	0.7	0.690	0.700	0.688

Table 5. Inclusion probabilities for some different λ_1 N=30												
n	λ_1 = 0.2				λ_1 = 0.5				λ_1 = 0.7			
	λ_1/λ_2	π_1(n;U)	π_1(n;E)	π_1(n;P)	λ_1/λ_2	π_1(n;U)	π_1(n;E)	π_1(n;P)	λ_1/λ_2	π_1(n;U)	π_1(n;E)	π_1(n;P)
1	7.3	0.239	0.216	0.204	29.0	0.629	0.579	0.538	67.7	0.813	0.800	0.766
2	3.2	0.215	0.206	0.201	9.7	0.578	0.530	0.508	15.6	0.751	0.729	0.712
3	2.1	0.207	0.203	0.200	5.8	0.557	0.517	0.504	8.8	0.741	0.716	0.705
4	1.5	0.204	0.201	0.200	4.1	0.544	0.512	0.502	6.2	0.736	0.710	0.703
5	1.2	0.201	0.201	0.200	3.2	0.534	0.508	0.501	4.7	0.733	0.707	0.702
6	1.0	0.200	0.200	0.200	2.6	0.526	0.506	0.501	3.8	0.730	0.705	0.701
7	0.9	0.199	0.200	0.200	2.2	0.520	0.505	0.501	3.2	0.728	0.704	0.701
8	0.7	0.198	0.199	0.200	1.9	0.515	0.504	0.500	2.8	0.725	0.703	0.701
9	0.7	0.198	0.199	0.200	1.7	0.512	0.503	0.500	2.4	0.723	0.703	0.700
10	0.6	0.197	0.199	0.200	1.5	0.509	0.502	0.500	2.2	0.721	0.702	0.700
11	0.5	0.197	0.198	0.200	1.4	0.506	0.502	0.500	2.0	0.718	0.702	0.700
12	0.5	0.197	0.198	0.200	1.3	0.504	0.501	0.500	1.8	0.716	0.701	0.700
13	0.5	0.196	0.198	0.200	1.2	0.503	0.501	0.500	1.7	0.714	0.701	0.700
14	0.4	0.196	0.198	0.200	1.1	0.501	0.500	0.500	1.5	0.711	0.701	0.700
15	0.4	0.196	0.198	0.200	1.0	0.500	0.500	0.500	1.4	0.709	0.701	0.700
16	0.4	0.196	0.197	0.200	0.9	0.499	0.500	0.500	1.3	0.707	0.701	0.700
17	0.3	0.196	0.197	0.200	0.9	0.498	0.499	0.500	1.2	0.706	0.700	0.700
18	0.3	0.196	0.197	0.200	0.8	0.497	0.499	0.500	1.2	0.704	0.700	0.700
19	0.3	0.195	0.197	0.200	0.8	0.496	0.499	0.500	1.1	0.703	0.700	0.700
20	0.3	0.195	0.197	0.200	0.7	0.496	0.499	0.500	1.1	0.701	0.700	0.700
21	0.3	0.195	0.196	0.200	0.7	0.495	0.498	0.499	1.0	0.700	0.700	0.700
22	0.3	0.195	0.196	0.200	0.7	0.495	0.498	0.499	1.0	0.699	0.700	0.700
23	0.3	0.195	0.195	0.199	0.6	0.494	0.497	0.499	0.9	0.698	0.700	0.700
24	0.2	0.195	0.195	0.199	0.6	0.494	0.497	0.499	0.9	0.697	0.700	0.700
25	0.2	0.195	0.194	0.199	0.6	0.493	0.497	0.499	0.8	0.696	0.700	0.700
26	0.2	0.195	0.193	0.198	0.6	0.493	0.496	0.498	0.8	0.695	0.700	0.699
27	0.2	0.195	0.191	0.196	0.5	0.492	0.495	0.496	0.8	0.695	0.700	0.699
28	0.2	0.195	0.186	0.190	0.5	0.492	0.492	0.492	0.7	0.694	0.700	0.697
29	0.2	0.195	0.161	0.130	0.5	0.492	0.480	0.462	0.7	0.693	0.700	0.688

5.3.4 Results for situations with fixed λ-ratio

Table 6. Inclusion probabilities for some different λ-ratios N=15												
n	λ_1/λ_2 = 3				λ_1/λ_2 = 2				λ_1/λ_2 = 0.2			
	λ_1	π_1(n;U)	π_1(n;E)	π_1(n;P)	λ_1	π_1(n;U)	π_1(n;E)	π_1(n;P)	λ_1	π_1(n;U)	π_1(n;E)	π_1(n;P)
1	0.176	0.200	0.186	0.179	0.125	0.133	0.129	0.126	0.014	0.013	0.014	0.014
2	0.353	0.396	0.368	0.357	0.250	0.267	0.257	0.251	0.028	0.027	0.027	0.028
3	0.529	0.580	0.545	0.533	0.375	0.399	0.383	0.377	0.042	0.040	0.041	0.042
4	0.706	0.738	0.714	0.708	0.500	0.530	0.508	0.502	0.056	0.053	0.055	0.056
5	0.882	0.857	0.876	0.883	0.625	0.656	0.631	0.627	0.070	0.067	0.068	0.070
6	-				0.750	0.769	0.752	0.751	0.085	0.080	0.082	0.084
7	-				0.875	0.862	0.870	0.875	0.099	0.093	0.095	0.098
8	-				-				0.113	0.107	0.108	0.112
9	-				-				0.127	0.120	0.122	0.126
10	-				-				0.141	0.133	0.134	0.140

Table 7. Inclusion probabilities for some different λ-ratios N=20

n	$\lambda_1/\lambda_2 = 4$				$\lambda_1/\lambda_2 = 2$				$\lambda_1/\lambda_2 = 0.2$			
	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$
1	0.174	0.199	0.184	0.176	0.095	0.100	0.097	0.096	0.010	0.010	0.010	0.010
2	0.348	0.395	0.364	0.352	0.190	0.200	0.194	0.191	0.021	0.020	0.020	0.021
3	0.522	0.576	0.538	0.526	0.286	0.300	0.291	0.287	0.031	0.030	0.031	0.031
4	0.696	0.731	0.705	0.698	0.381	0.400	0.387	0.382	0.042	0.040	0.041	0.042
5	0.870	0.848	0.865	0.870	0.476	0.499	0.482	0.477	0.052	0.050	0.051	0.052
6	-				0.571	0.597	0.577	0.572	0.063	0.060	0.061	0.062
7	-				0.667	0.691	0.670	0.667	0.073	0.070	0.071	0.073
8	-				0.762	0.778	0.763	0.762	0.083	0.080	0.081	0.083
9	-				0.857	0.853	0.854	0.857	0.094	0.090	0.091	0.094
10	-				0.952	0.912	0.945	0.952	0.104	0.100	0.101	0.104

Table 8. Inclusion probabilities for some different λ-ratios N=25

n	$\lambda_1/\lambda_2 = 4$				$\lambda_1/\lambda_2 = 2$				$\lambda_1/\lambda_2 = 0.2$			
	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$
1	0.143	0.160	0.150	0.144	0.077	0.080	0.078	0.077	0.008	0.008	0.008	0.008
2	0.286	0.319	0.298	0.288	0.154	0.160	0.157	0.154	0.017	0.016	0.016	0.017
3	0.429	0.474	0.442	0.431	0.231	0.240	0.234	0.231	0.025	0.024	0.024	0.025
4	0.571	0.618	0.584	0.574	0.308	0.320	0.312	0.308	0.033	0.032	0.032	0.033
5	0.714	0.744	0.721	0.716	0.385	0.400	0.390	0.385	0.041	0.040	0.041	0.041
6	0.857	0.844	0.854	0.858	0.462	0.480	0.467	0.462	0.050	0.048	0.049	0.050
7	-				0.538	0.559	0.543	0.539	0.058	0.056	0.057	0.058
8	-				0.615	0.637	0.619	0.616	0.066	0.064	0.065	0.066
9	-				0.692	0.713	0.695	0.693	0.074	0.072	0.073	0.074
10	-				0.769	0.784	0.770	0.770	0.083	0.080	0.081	0.083

Table 9. Inclusion probabilities for some different λ-ratios N=30

n	$\lambda_1/\lambda_2 = 5$				$\lambda_1/\lambda_2 = 2$				$\lambda_1/\lambda_2 = 0.2$			
	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$
1	0.147	0.166	0.155	0.149	0.065	0.067	0.066	0.065	0.007	0.007	0.007	0.007
2	0.294	0.331	0.307	0.297	0.129	0.133	0.131	0.129	0.014	0.013	0.014	0.014
3	0.441	0.491	0.456	0.444	0.194	0.200	0.196	0.194	0.021	0.020	0.020	0.021
4	0.588	0.637	0.601	0.591	0.258	0.267	0.261	0.258	0.027	0.027	0.027	0.027
5	0.735	0.761	0.741	0.737	0.323	0.333	0.326	0.323	0.034	0.033	0.034	0.034
6	0.882	0.856	0.877	0.883	0.387	0.400	0.391	0.388	0.041	0.040	0.040	0.041
7	-				0.452	0.467	0.456	0.452	0.048	0.047	0.047	0.048
8	-				0.516	0.533	0.520	0.517	0.055	0.053	0.054	0.055
9	-				0.581	0.4599	0.584	0.581	0.062	0.060	0.061	0.062
10	-				0.645	0.664	0.648	0.646	0.068	0.067	0.067	0.068

Table 10. Inclusion probabilities for some different λ-ratios N=50												
n	$\lambda_1/\lambda_2 = 8$				$\lambda_1/\lambda_2 = 3$				$\lambda_1/\lambda_2 = 0.2$			
	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$
1	0.140	0.160	0.148	0.142	0.058	0.060	0.059	0.058	0.004	0.004	0.004	0.004
2	0.281	0.318	0.294	0.283	0.115	0.120	0.117	0.116	0.008	0.008	0.008	0.008
3	0.421	0.471	0.436	0.424	0.173	0.180	0.176	0.173	0.012	0.012	0.012	0.012
4	0.561	0.613	0.575	0.564	0.231	0.240	0.234	0.231	0.016	0.016	0.016	0.016
5	0.702	0.736	0.710	0.703	0.288	0.300	0.293	0.289	0.020	0.020	0.020	0.020
6	0.842	0.833	0.840	0.843	0.346	0.360	0.351	0.347	0.024	0.024	0.024	0.024
7	0.982	0.903	0.973	0.982	0.404	0.420	0.409	0.404	0.028	0.028	0.028	0.028
8	-				0.462	0.480	0.466	0.462	0.033	0.033	0.033	0.033
9	-				0.519	0.540	0.524	0.520	0.037	0.037	0.037	0.037
10	-				0.577	0.599	0.581	0.577	0.041	0.041	0.041	0.041

Table 11. Inclusion probabilities for some different λ-ratios N=100												
n	$\lambda_1/\lambda_2 = 8$				$\lambda_1/\lambda_2 = 3$				$\lambda_1/\lambda_2 = 0.2$			
	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$	λ_1	$\pi_1(n;U)$	$\pi_1(n;E)$	$\pi_1(n;P)$
1	0.075	0.080	0.077	0.075	0.029	0.030	0.030	0.029	0.002	0.002	0.002	0.002
2	0.150	0.160	0.154	0.150	0.059	0.060	0.059	0.059	0.004	0.004	0.004	0.004
3	0.224	0.240	0.230	0.225	0.088	0.090	0.089	0.088	0.006	0.006	0.006	0.006
4	0.299	0.320	0.306	0.300	0.118	0.120	0.119	0.118	0.008	0.008	0.008	0.008
5	0.374	0.400	0.382	0.375	0.147	0.150	0.148	0.147	0.010	0.010	0.010	0.010
6	0.449	0.479	0.456	0.449	0.176	0.180	0.178	0.177	0.012	0.012	0.012	0.012
7	0.523				0.206	0.210	0.208	0.206	0.014	0.014	0.014	0.014
8	0.598				0.235	0.240	0.237	0.235	0.016	0.016	0.016	0.016

References

Loève, M. (1960). *Probability Theory*. D. van Nostrand Co., Princeton.

Ohlsson, E. (1990). Sequential Poisson Sampling from a Business Register and Its Application to the Swedish Consumer Price Index. Statistics Sweden R&D Report 1990:6.

Rosén, B. (1997a). Asymptotic Theory for Order Sampling. *J. Statist. Planning and Inf.*, **62** 135-158.

Rosén, B. (1997b). On Sampling with Probability Proportional to Size. *J Stat. Plan. Inf.*, **62** 159-191.

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