R&D Report 2000:2
Research – Methods – Development

Statistiska centralbyrån Statistics Sweden

On Inclusion Probabilities and Estimator Bias for Pareto πps Sampling

Nibia Aires & Bengt Rosén
Chalmers University of Technology & Statistics Sweden

INLEDNING

TILL

R & D report : research, methods, development / Statistics Sweden. – Stockholm : Statistiska centralbyrån, 1988-2004. – Nr. 1988:1-2004:2.
Häri ingår Abstracts : sammanfattningar av metodrapporter från SCB med egen numrering.

Föregångare:

Metodinformation : preliminär rapport från Statistiska centralbyrån. – Stockholm : Statistiska centralbyrån. – 1984-1986. – Nr 1984:1-1986:8.

U/ADB / Statistics Sweden. – Stockholm : Statistiska centralbyrån, 1986-1987. – Nr E24-E26

R & D report : research, methods, development, U/STM / Statistics Sweden. – Stockholm : Statistiska centralbyrån, 1987. – Nr 29-41.

Efterföljare:

Research and development : methodology reports from Statistics Sweden. – Stockholm : Statistiska centralbyrån. – 2006-. – Nr 2006:1-.

R & D Report 2000:2. On inclusion probabilities and estimator bias for Pareto πps sampling / Nibia Aires; Bengt Rosén.
Digitaliserad av Statistiska centralbyrån (SCB) 2016.

urn:nbn:se:scb-2000-X101OP0002

R&D Report **2000:2**
Research – Methods – Development

On Inclusion Probabilities and Estimator Bias for Pareto πps Sampling

Nibia Aires & Bengt Rosén
Chalmers University of Technology & Statistics Sweden

R&D Report 2000:2
Research - Methods - Development

On Inclusion Probabilities and Estimator Bias for Pareto πps Sampling

---

Från trycket	Mars 2000
Producent	Statistiska centralbyrån, *Statistics Sweden,* metodenheten Box 24300, *SE*-104 51 STOCKHOLM
Utgivare	Lars Lyberg

Förfrågningar	Bengt Rosén bengt.rosen@scb.se telefon 08-506 944 90 telefax 08-506 977 88

ISSN 0283-8680

Printed in Sweden
SCB-Tryck, Örebro 2000

March 2000

On Inclusion Probabilities and Estimator Bias for Pareto πps Sampling

Nibia Aires & Bengt Rosén

Chalmers University of Technology & Statistics Sweden

Abstract

A means for utilizing auxiliary information in surveys is to sample with inclusion probabilities proportional to given size values, to use a πps design, preferably with fixed sample size. A candidate in that context is Pareto πps. This scheme has a number of attractive properties, notably simple sample selection, good resulting estimation accuracy, simple variance estimation and simple procedures for coordination of samples by permanent random numbers.

However, Pareto πps was derived by limit considerations and works with some degree of approximation for finite samples. In particular, desired and factual inclusion probabilities do not agree exactly, which in turn leads to some estimator bias. Practically useful information on small sample behavior of Pareto πps can, to the best of our understanding, only be gained by numerical studies. Earlier investigations with that purpose have been too limited to allow for general conclusions, while this paper reports on a very extensive numerical study. The main conclusion is that estimator bias is negligible in almost all situations met in survey practice.

CONTENTS

On inclusion probabilities and estimator bias for Pareto πps sampling

1 Introduction and outline

A means for utilizing auxiliary information in surveys is to sample with inclusion probabilities proportional to given size values, to use a πps design, preferably with fixed sample size. A candidate in that context is ***Pareto πps***, introduced independently by Rosén (1997) and Saavedra (1995). This scheme has, as accounted for in Rosén (1997), many attractive properties, notably simple sample selection, good estimation accuracy, simple variance estimation and simple procedures for coordination of samples by permanent random numbers.

Pareto πps was derived by limit considerations, and works with some approximation. In particular, desired and factual inclusion probabilities do not agree exactly. Rosén (2000) proved, though, that they under very general conditions are asymptotically (as the sample size tends to infinity) equal. Numerical investigations by Rosén (2000) and Aires (1999, 2000) indicated that the convergence is rapid. These studies were too limited, though, to allow for general conclusions on how well desired inclusion probabilities are approximated by the factual ones. This paper reports on a much more extensive numerical study, in which the chief tool has been the algorithm in Aires (1999) for computation of Pareto πps inclusion probabilities.

The problem of how well desired inclusion probabilities are approximated has per se mainly theoretical interest. However, as is emphasized in the following, there is close connection between approximation accuracy for inclusion probabilities and estimator bias, the latter being an issue of great practical relevance. The convergence of inclusion probabilities implies that estimator bias is asymptotically negligible. Its magnitude for finite samples has been an open question, though. The chief aim in this paper is to enlighten this problem. The main conclusion is, somewhat sweepingly formulated, that the bias is negligible in practical survey situations.

The paper is organized as follows. Sections 2 and 3 are expository and review some basics on πps sampling in general respectively on Pareto πps. Measures of approximation accuracy for inclusion probabilities and estimator bias are introduced in Section 4. Section 5 specifies certain size value patterns which play a distinguished role in the numerical study. The detailed numerical findings are presented in Appendices 1 and 2, containing tables and graphs respectively. Recommendations for practical use of Pareto πps are formulated in Section 6.

2 Generalities on πps sampling

We consider probability sampling without replacement with fixed sample size from a population $U = (1,2,...,N)$, on which a study variable $y = (y_1, y_2, ..., y_N)$ is defined. A frame which one-to-one corresponds with the population units is available. It is presumed that the frame contains unit-wise auxiliary information $s = (s_1, s_2, ..., s_N)$, $s_i > 0$, $i \in U$, interpreted as ***size values*** which typically are positively correlated with the study variable.

A sampling design is a ***strict πps scheme*** if its factual inclusion probabilities π_i, $i \in U$, are the following ***desired inclusion probabilities*** $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$, n standing for sample size;

$$\lambda_i = n \cdot s_i \Big/ \sum_{j=1}^{N} s_j \,, \quad i = 1,2,...,N. \tag{2.1}$$

Remark 2.1 : Formula (2.1) can lead to λ-values exceeding 1, which is incompatible with being probabilities. If so, the usual "adjustment" is to assign the units with largest size values to a "sample for certain" stratum. A πps sample is then drawn from the remaining units (with remaining sample size). In the sequel is presumed that $0 < \lambda_i < 1$, $i \in U$. ¤

As stated, a strict πps scheme is characterized by the relation $\pi_i = \lambda_i$, $i \in U$. We will be more generous, though, and accept a sampling scheme as a ***πps scheme*** if (2.2) below is met;

$$\pi_i \approx \lambda_i \text{ holds with good approximation for } i = 1, 2, \ldots, N. \tag{2.2}$$

In the strict πps case, the Horvitz-Thompson (HT) estimator for a total $\tau(y) = y_1 + y_2 + \ldots + y_N$ is as stated below. As is well known, this estimator is unbiased.

$$\hat{\tau}(y)_{HT} = \sum_{i \in \text{Sample}} y_i / \lambda_i . \tag{2.3}$$

We presume that the estimator in (2.3) is used also under the more generous πps notion based on (2.2). Then it may have some bias, though. The estimator is re-stated in (2.4) where it is denoted $\hat{\tau}(y)$, which henceforth stands for this particular estimator. In (2.4) it is also written on an alternative form, which will be useful a bit later on. $I_1, I_2, \ldots, I_N$ denote the sample inclusion indicators, i.e. $I_i = 1$ if unit i is selected to the sample and $= 0$ otherwise.

$$\hat{\tau}(y) = \sum_{i \in \text{Sample}} y_i / \lambda_i = \sum_{i=1}^{N} (y_i / \lambda_i) \cdot I_i . \tag{2.4}$$

3 On Pareto πps

3.1 Definition

DEFINITION 3.1: The ***Pareto πps scheme*** with ***size values*** $\mathbf{s} = (s_1, s_2, \ldots, s_N)$ and ***sample size*** n generates a sample by the following steps.

1. The desired inclusion probabilities $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ are computed by (2.1).
2. Independent random variables $R_1, R_2, \ldots, R_N$ with uniform distribution on $[0, 1]$ are realized, and ***ranking variables*** Q are computed as follows;

$$Q_i = \frac{R_i \cdot (1 - \lambda_i)}{\lambda_i \cdot (1 - R_i)}, \quad i = 1, 2, \ldots, N. \tag{3.1}$$

3. The ***sample*** consists of the units with the ***n smallest Q-values.***

It is by no means obvious that the above scheme actually is a πps scheme (in the (2.2) sense). However, Rosén (2000) proved that (2.2) holds with asymptotic (as $n \to \infty$) equality.

As stated earlier, a main task for the present study was to find out how well approximation (2.2) works for finite Pareto πps samples. The central measure of approximation goodness will be the ***maximal absolute relative error*** (for inclusion probabilities);

$$\Psi = \max_{i=1,2,\ldots,N} |\pi_i / \lambda_i - 1| . \tag{3.2}$$

Ψ is a natural performance measure in the approximation problem, which is rather theoretical, though. However, Ψ also has considerable practical interest due to fact that there is close connection between Ψ and the magnitude of estimator bias, which is discussed next.

3.2 On estimator bias for Pareto πps

We presume that the sample is drawn by Pareto πps and that the estimator $\hat{\tau}(y)$ in (2.4) is used. As in Section 2, we confine the estimation considerations to the "fundamental" problem, estimation of a population total. Since (2.2) holds, $\hat{\tau}(y)$ is afflicted with some bias. To get an expression for it, we take expectation in (2.4);

$$E[\hat{\tau}(y)] = \sum_{i=1}^{N} (y_i / \lambda_i) \cdot E(I_i) = \sum_{i=1}^{N} y_i \cdot (\pi_i / \lambda_i) = \tau(y) + \sum_{i=1}^{N} y_i \cdot (\pi_i / \lambda_i - 1) . \tag{3.3}$$

Hence;

The ***bias for the estimator*** $\hat{\tau}(y)$ is $E[\hat{\tau}(y)] - \tau(y) = \sum_{i=1}^{N} y_i \cdot (\pi_i/\lambda_i - 1)$. (3.4)

Formulas (3.4) and (3.2) yield;

The ***absolute relative bias for the estimator*** $\hat{\tau}(y)$ is

$$\left|[E[\hat{\tau}(y)] - \tau(y)]/\tau(y)\right| = \left|\sum_{i=1}^{N} y_i \cdot (\pi_i/\lambda_i - 1)\Big/\tau(y)\right| \leq \Psi \cdot \left(\sum_{i=1}^{N} |y_i|\Big/\tau(y)\right). \quad (3.5)$$

If the study variable y takes only non-negative values, as is the case in most practical surveys, the last factor in (3.5) equals 1. Hence;

For a non-negative study variable y: The absolute relative bias for $\hat{\tau}(y) \leq \Psi$. (3.6)

Remark 3.1: The bounds in (3.5) and (3.6) are often conservative for the following reasons. (i) They disregard cancellation effects due to alternating signs of $\pi_i/\lambda_i - 1$. (ii) All discrepancies $|\pi_i/\lambda_i - 1|$ do not have the maximal value Ψ. The bounds can be attained, though, e.g. with $y_i = 0$ for i with $|\pi_i/\lambda_i - 1| < \Psi$, and $y_i = \text{sign}(\pi_i/\lambda_i - 1)$ for i with $|\pi_i/\lambda_i - 1| = \Psi$. ¤

3.3 Chief questions in the numerical study

A pair (N ; **s**) of a population size N and size values $\mathbf{s} = (s_1, s_2, ..., s_N)$ is referred to as a ***size value situation***. A Pareto πps scheme is specified by (N, **s**) and the sample size n. When we want to emphasize dependence on one or more of these parameters, we use notation as Pareto πps(N, **s**, n) or Pareto πps(**s**), $\lambda_i(n)$, $\lambda_i(n;\mathbf{s})$, $\lambda_i(n;N;\mathbf{s})$ and $\pi_i(n)$, $\pi_i(n;\mathbf{s})$, $\pi_i(n;N;\mathbf{s})$ for desired and factual inclusion probabilities. Analogously, Ψ in (3.2) is often elaborated to;

$$\Psi(n;N;\mathbf{s}) = \max_i |\pi_i(n;N;\mathbf{s})/\lambda_i(n;N;\mathbf{s}) - 1|. \quad (3.7)$$

The chief problems that are addressed are stated in (3.8) and (3.9) below. Note that (3.9) is a "converse" to (3.8).

For a specific size value situation (N; **s**):

How large, at most, is $\Psi(n;N;\mathbf{s})$ for a specific sample size n? (3.8)

Which sample sizes n imply $\Psi(n;N;\mathbf{s}) \leq \beta$ for a specified $\beta > 0$? (3.9)

In the first round questions (3.8) and (3.9) relate to the approximation (2.2), how well the factual inclusion probabilities approximate the desired ones. However, by virtue of (3.5) and (3.6), answers to (3.8) and (3.9) also provide information on relative estimator bias. In the sequel ***the approximation problem*** refers to both these aspects, estimator bias as well as discrepancy between factual and desired inclusion probabilities.

To the best of our understanding it is in vain to hope for precise answers to (3.8) and (3.9) via analytical formulas. One has to be content with (fairly) coarse answers derived by numerically demanding computation efforts. The employed numerical algorithm is described next.

3.4 The computation algorithm

The chief work in deriving answers to (3.5) and (3.6) consisted of computation of $\pi_i(n;N;\mathbf{s})$, $i = 1, 2, ..., N$, for a rich set of values for (N, **s**) and n. For that the core tool was the algorithm for computation of Pareto πps inclusion probabilities which is derived and justified in Aires (1999). To give an idea of the numerical efforts, a sketch of the algorithm is presented below. It describes computation of $\pi_i(n;N;\mathbf{s})$ for i = N. π_i-value for general i were computed by appropriate re-labeling of the population units.

Computation algorithm for Pareto inclusion probabilities

The given quantities are N, s and n.

Step 1: Compute $\lambda_i(n,\mathbf{s})$ by (2.1), $i = 1,2,\dots,N$.

Step 2: For a mesh $\mathcal{M}$ of t-values, which is fine enough to yield desired precision in the numerical integration in Step 3, compute $\{F_n^{N-1}(t) : t \in \mathcal{M}\}$ by the double recursion;

$$F_\nu^M(t) = F_\nu^{M-1}(t)\cdot\frac{1-\lambda_\nu}{1+\lambda_\nu(t-1)} + F_{\nu-1}^{M-1}(t)\cdot\frac{\lambda_\nu\cdot t}{1+\lambda_\nu(t-1)},$$
$$\nu = 1,2,\dots,n,\quad M = 1,2,\dots,N, \tag{3.10}$$

with boundary conditions

$$F_0^M(t) = 1,\ \text{for all } M \text{ and } 0 \le t < \infty. \tag{3.11}$$

Step 3: Compute, by numerical integration;

$$\pi_N(n;N,\mathbf{s}) = \lambda_N\cdot(1-\lambda_N)\cdot\int_0^\infty (1-F_n^{N-1}(t))/[1+\lambda_N\cdot(t-1)]^2\,dt. \tag{3.12}$$

4 Bounds employed in the approximation problem

4.1 Some definitions

In the sequel size measures $\mathbf{s}=(s_1, s_2, \dots, s_N)$ are presumed to be ***normed*** so that average size is 1, i.e. so that (4.1) below holds;

$$\frac{1}{N}\cdot\sum_{i=1}^{N} s_i = 1. \tag{4.1}$$

A normed **s** is called a ***size value pattern***. Set;

$$s_{min} = \min\{s_i : i = 1,2,\dots,N\},\quad s_{max} = \max\{s_i : i = 1,2,\dots,N\}. \tag{4.2}$$

As stated in Remark 1.1, it is presumed that all λ_i given by (2.1) are smaller than 1. This lays the following constraint on the sample size n, where $[\cdot]$ denotes integral part and - "less than";

$$n \le n_m = n_m(N;\mathbf{s}) := [N/s_{max}-]. \tag{4.3}$$

The quantity n_m in (4.3) is called the ***maximal sample size*** in situation $(N;\mathbf{s})$. An n which satisfies (4.3) is said to be an ***admissible sample size*** in situation $(N;\mathbf{s})$.

λ - values close to 1 may lead to "capricious" samples, which can be avoided by prescribing that $\lambda_i \le \alpha$, $i = 1,2,\dots,N$, for some specified $\alpha < 1$. $\alpha = 0.9, 0.8$ are considered in the numerical context. The ***α-maximal sample size*** $n_{m,\alpha}$ and ***α-admissible sample sizes*** are determined by;

$$n \le n_{m,\alpha}(N;\mathbf{s}) := [\alpha\cdot N/s_{max}]. \tag{4.4}$$

In Section 6 $n_{m,\alpha}$ is also used as a means for stating conditions to the effect that a sample size must not be "too large". For that purpose also $\alpha = 0.5$ was considered.

We shall relate approximation error bounds to ***size pattern families*** of the following type;

$$\mathcal{S}(N;\gamma,\delta) = \{\mathbf{s}: \text{population size} = N,\ s_{min} = \gamma,\ s_{max} = \delta\},\ 0 < \gamma \le 1 \le \delta < \infty. \tag{4.5}$$

In words: A size pattern is in $\mathcal{S}(N;\gamma,\delta)$ if at least one population unit has (normed) size value γ, at least one has value δ, and the others have size values in $[\gamma,\delta]$. This kind of family is of interest for at least the following reasons. (i) When all size values are equal, $\gamma = \delta = 1$, Pareto πps(**s**) is nothing but simple random sampling, with $\pi_i(n) = \lambda_i(n) = n/N$, and the approximation (2.2) is perfect. Thus, for an approximation problem to be at hand, different size values must occur. $\mathcal{S}(N;\gamma,\delta)$ lays constraints on *how different* they can be. The smaller γ and the larger δ

is, the more different are size values. (ii) It is simple to determine to which $\mathcal{S}(N;\gamma,\delta)$ a size pattern belongs by computing the smallest and largest normed size values.

Since $n_m(N;s)$ and $n_{m,\alpha}(N;s)$ in (4.3) and (4.4) depend only on N and s_{max}, they are the same for all patterns in $\mathcal{S}(N;\gamma,\delta)$. We therefore use the following simpler notation;

For $s \in \mathcal{S}(N;\gamma,\delta)$ we write $n_m(N;\delta)$ and $n_{m,\alpha}(N;\delta)$ for $n_m(N;s)$ and $n_{m,\alpha}(N;s)$. (4.6)

4.2 Bounding sequences

A size value situation (N; s) determines a sequence $\{\Psi(n;N;s): n=1,2,\dots,n_m\}$, with Ψ given by (2.4), which we refer to as the associated ***Ψ-sequence***. Such sequences will play a central role in the subsequent considerations.

At the outset of this study we had various conjectures about the behavior of Ψ - sequences. Many of these turned out to be wrong when confronted with numerical data. One was that all Ψ - sequences have bath-tub shape. That this is not true in general is illustrated in Figure 4.1, which shows Ψ - sequences for three different size value patterns, all with $N = 100$, $\gamma = 0.5$ and $\delta = 2$. Their names, "boundary", "middle" and "even" are explained later on. A multitude of other Ψ - sequence graphs are presented in Appendix 2.

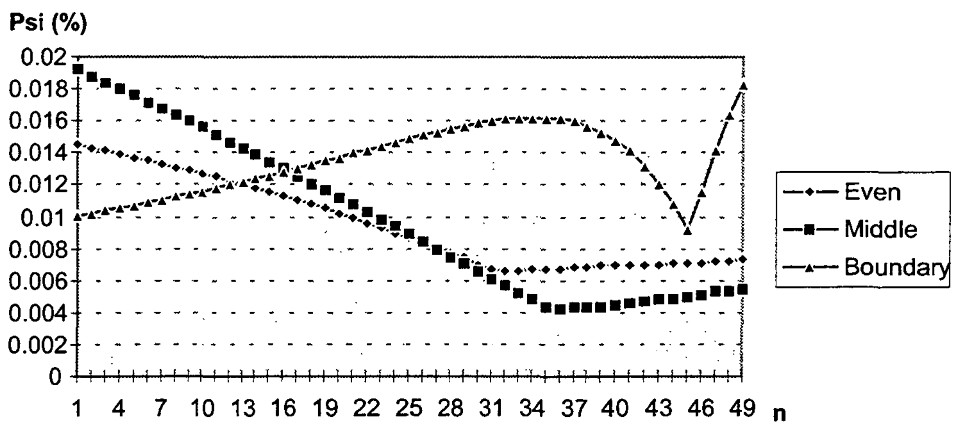


Figure 4.1. Psi sequences for three size patterns

Our aim is to answer (3.8) and (3.9) in terms of the parameters N, γ and δ. Figure 4.1 shows that there is no simple domination rule for Ψ - sequences for different size value patterns in the same family $\mathcal{S}(N;\gamma,\delta)$. As functions of n they can take turn to lie above each other. To find bounds which hold uniformly for N, γ and δ we must introduce envelope notions.

4.2.1 Ψ-envelope sequences

The ***Ψ-envelope sequence for the family*** $\mathcal{S}(N;\gamma,\delta)$, denoted $\Psi^*(\cdot;N;\gamma;\delta)$, is (recall (4.6);

$$\Psi^*(n;N;\delta;\gamma) = \sup_{s\in\mathcal{S}(N;\gamma,\delta)} \Psi(n;N;s), \quad n=1,2,\dots,n_m(N;\delta). \tag{4.7}$$

In words $\Psi^*(n;N;\delta;\gamma)$ means the maximal relative approximation error in (2.2) for a Pareto πps sample of size n selected from a population of size N with maximal and minimal normed size values δ and γ. Technically formulated;

For $s \in \mathcal{S}(N;\gamma,\delta)$ and $n \le n_m(N;\delta)$:

$$\left|\pi_i(n;s)/\lambda_i(n;s) - 1\right| \le \Psi^*(n;N;\gamma;\delta), \quad i=1,2,\dots,N. \tag{4.8}$$

Hence, knowledge of $\Psi^*(\cdot;N;\gamma;\delta)$ enables answers to (3.8). In fact, Ψ^* is the smallest upper bound sequence that works for all **s** in $\mathcal{S}(N;\gamma,\delta)$.

4.2.2 Quasi envelopes

The envelope (4.7) is defined in terms of suprema over the *infinite* family $\mathcal{S}(N;\gamma,\delta)$. To compute it in practice, one needs to know a *finite **extremal sub-family*** of $\mathcal{S}(N;\gamma,\delta)$, by which we mean a finite sub-family with the same envelope. Regrettably, we cannot exhibit such a sub-family with mathematical rigor. However, we strongly believe, supported by numerical findings, that the following intuitive arguments lead to a "close to extremal" sub-family.

Numerically extremal size value patterns (as regards Ψ-values) are found among geometrically extremal patterns. In the latter category, the following three types of size value patterns come into mind. (i) Patterns with size values (fairly) ***evenly spread*** over $[\gamma,\delta]$. (ii) Patterns with the majority of size values in the ***middle*** of $[\gamma,\delta]$. (iii) Patterns with the majority of size values at the ***boundaries*** of $[\gamma,\delta]$. Precise specifications of such patterns are given in Section 5, where they are denoted $\mathbf{s}(N;\gamma;\delta;\mathbf{e})$, $\mathbf{s}(N;\gamma;\delta;\mathbf{m})$ and $\mathbf{s}(N;\gamma;\delta;\mathbf{b})$, **e** for "even spread", **m** for "middle" and **b** for "boundary".

The ***quasi Ψ- envelope sequence for*** $\mathcal{S}(N;\gamma,\delta)$, denoted $\Psi^{**}(\cdot;N;\gamma;\delta)$ is;

$$\Psi^{**}(n;N;\gamma;\delta)=\max\{\Psi(n;\mathbf{s}(N;\gamma;\delta;\mathbf{e})),\Psi(n;\mathbf{s}(N;\gamma;\delta;\mathbf{m})),\Psi(n;\mathbf{s}(N;\gamma;\delta;\mathbf{b}))\},$$
$$n=1,2,\ldots,n_m(N;\delta). \quad (4.9)$$

Believing that $\{s(\mathbf{e}), s(\mathbf{m}), s(\mathbf{b})\}$ is a "close to extremal" sub-family of $\mathcal{S}(N;\gamma,\delta)$ we work under the following presumption in the sequel;

$\Psi^{**}(\cdot;N;\gamma;\delta)$ yields good approximation of the true envelope $\Psi^*(\cdot;N;\gamma;\delta)$. (4.10)

Since Ψ^{**} is determined by just three size value patterns it is computable provided that a computation algorithm for $\Psi(n;N;\mathbf{s})$ for given **s** and n is available, which it is by Section 3.4. Strictly mathematically, though, (4.10) is a conjecture, based on intuition and with some numerical support. We made attempts to justify (4.10) more rigorously by employing numerical optimization programs to find extremal size value patterns. However, this approach turned out to be unfeasible at least with the optimization programs we tried.

4.2.3 Upper sequences

A quasi envelope may be quite irregular, wiggling up and down, as seen from Figure 4.1. To enable simple answers to (3.8) and (3.9) we introduce coarser upper bound sequences (than the quasi envelope), called upper sequences, which are non-increasing functions of sample size.

The ***upper sequence*** $\Gamma(\cdot)$ ***for the family*** $\mathcal{S}(N;\gamma,\delta)$ is;

$$\Gamma(n_0;N;\gamma;\delta)=\max\{\Psi^{**}(n;N;\gamma;\delta): n_0\le n\le n_m(N;\delta)\},\ n_0=1,2,\ldots,n_m(N;\delta). \quad (4.11)$$

In words: $\Gamma(n_0;N;\gamma;\delta)$ bounds the relative approximation error in (2.2) for a Pareto πps sample from a population of size N, with maximal and minimal normed size values δ and γ, for all sample sizes $\ge n_0$.

Under (4.10) the following bound holds with good approximation.

For $\mathbf{s}\in\mathcal{S}(N;\gamma,\delta)$;

$$\Psi(n;N;\mathbf{s})\le\Gamma(n_0;N;\gamma;\delta),\ n_0\le n\le n_m(N;\delta),\ n_0=1,2,\ldots,n_m(N;\delta). \quad (4.12)$$

4.2.4 Upper sequences for α-admissible sample sizes

Since Pareto πps is based on limit considerations, one believes in the first round that conditions for good approximation basically should be of the type "provided the sample size is *at least* ...". However, as seen from the graphs in Appendix 2, $\Psi^{**}(\cdot)$ often attains its largest values for large (admissible) sample sizes. As a consequence, sharp conditions for good approximation must also contain an ingredient of the type "provided the sample size is *at most* ...". This aspect will technically be handled as follows. An α is specified, $0<\alpha<1$, and used in conditions saying that sample sizes must not exceed $n_{m,\alpha}$ in (4.4), which is a way of saying "provided the sample size is *at most* ...". In line with this, the notion of upper sequence is extended as follows.

The ***upper sequence for α-admissible sample sizes*** is

$$\Gamma_\alpha(n_0;N;\gamma;\delta)=\max\{\Psi^{**}(n;N;\gamma;\delta): n_0\le n\le n_{m,\alpha}(N;\delta)\},\ n_0=1,2,\dots,n_{m,\alpha}(N;\delta). \quad (4.13)$$

Under (4.10) the following bound holds with good approximation.

For $s\in\mathcal{S}(N;\gamma;\delta)$;

$$\Psi(n;N;s)\le\Gamma_\alpha(n_0;N;\gamma;\delta),\ n_0\le n\le n_{m,\alpha}(N;\delta),\ n_0=1,2,\dots,n_{m,\alpha}(N;\delta). \quad (4.14)$$

Remark 4.1: The maximum operations in (4.11) and (4.13) add to what is said in Remark 3.1. In (4.12) and (4.14) $\Gamma(n;N;\gamma;\delta)$ and $\Gamma_\alpha(n;N;\gamma;\delta)$ may be quite conservative bounds for many sample sizes n. ¤

Appendix 1 presents numerical values for general upper sequences as well as for α-admissible sample sizes, with $\alpha=0.9,\ 0.8,\ 0.5$.

4.2.5 Sufficient sample sizes

Let β, $0<\beta<1$, be a specified tolerance level for the relative approximation error in (2.2). By disregarding the (mildly) approximate nature of the statement in (4.10), an answer to (3.9) is given by the smallest n_0 such that $\Gamma(n_0;N;\gamma;\delta)\le\beta$, called the ***$\beta$-sufficient sample size*** for $\mathcal{S}(N;\gamma,\delta)$ and denoted by $n_0(\beta)$. It informs about sample sizes which are large enough to guarantee approximation accuracy β. Formally;

$$n_0(\beta)=\min\{n:\Gamma(n;N;\gamma;\delta)\le\beta\}. \quad (4.15)$$

As discussed in Section 4.2.4, large sample sizes rather than small ones jeopardize good approximation accuracy. This fact is addressed by the following notion. The ***(β,α)-sufficient sample size*** for $\mathcal{S}(N;\gamma,\delta)$, denoted $n_0(\beta,\alpha)$, is the smallest sample size which guarantees that $\Psi(n;N;\gamma;\delta)\le\beta$ for $n_0(\beta;\alpha)\le n\le n_{m,\alpha}(N;\delta)$. Formally;

$$n_0(\beta;\alpha)=\min\{n:\Gamma_\alpha(n;N;\gamma;\delta)\le\beta\}. \quad (4.16)$$

The set of n-values over which minimum is taken In (4.15) and (4.16) may be empty. Then n_0 is set to ***none***. Numerical β- and (β,α)-sufficient sample sizes are presented in Appendix 1.

4.2.6 Approximation accuracy and population size

A conjecture about Ψ-sequence behavior which was supported by the numerical findings is formulated in (4.17). Somewhat sweepingly formulated it says that approximation accuracy improves as population size increases. Still, also (4.17) is a conjecture without a strict mathematical proof. It can be checked numerically in Appendices 1 and 2, though.

For fixed n, γ, δ and α the values of upper sequences $\Gamma(n;N;\gamma;\delta)$ and $\Gamma_\alpha(n;N;\gamma;\delta)$ decrease as the population size N increases. (4.17)

4.2.7 Weak quasi envelopes

The quasi envelope in (4.9) dominates Ψ - sequences for all size pattern shapes. As illustrated by the graphs in Appendix 2, the "worst possible" pattern shape is the boundary pattern, unless when sample size is very small. Its Ψ - values mostly lie way above those for even spread and middle when sample size is "non - small". However, patterns of boundary type are unusual in practice where, at least we think so, most pattern shapes resemble even spread. Patterns in the last category will be said to lie ***in the vicinity of even spread***. With this background we introduce the ***weak quasi envelope*** $w\Psi^{**}$, which takes into account only the even spread and middle patterns;

$$w\Psi^{**}(n) = \max\{\Psi(n;s(N;\gamma;\delta;\mathbf{e})), \Psi(n;s(N;\gamma;\delta;\mathbf{m}))\}, \quad n = 1,2,\ldots,n_m(N;\delta). \tag{4.18}$$

Moreover, ***weak upper sequences***, ***weakly β - sufficient sample sizes*** and ***weakly (β, α) - sufficient sample sizes*** are defined in analogy with (4.11), (4.13) and (4.15) - (4.16), by using the weak quasi envelope $w\Psi^{**}$ instead of Ψ^{**}. Numerical values are presented in Appendix 1.

Remark 4.2: In Section 4.2.4 is pointed at the circumstance that the (full) quasi envelope usually attains its largest values for large sample sizes. This growth depends mainly on contribution from size patterns of boundary type. The weak quasi envelope, which is not influenced by boundary type patterns, behaves as "expected", it decreases as sample size increases.

5 The special size patterns

Here we give precise specifications of the size value patterns **s** in $\mathcal{S}(N;\gamma,\delta)$ which are mentioned in Section 4.2.2, boundary, middle and even patterns. Recall that (4.1) is presumed to hold and the relation;

$$\gamma = s_{min} \le s_i \le s_{max} = \delta, \quad i = 1,2,\ldots,N. \tag{5.1}$$

Set;

$$M_\gamma = N \cdot (\delta - 1)/(\delta - \gamma), \qquad M_\delta = N \cdot (1 - \gamma)/(\delta - \gamma). \tag{5.2}$$

Note the following relations;

$$M_\gamma + M_\delta = N, \quad \gamma \cdot M_\gamma + \delta \cdot M_\delta = N. \tag{5.3}$$

M_γ and M_δ split into integral parts, N_γ and N_δ, and fractional parts, F_γ and F_δ as follows;

$$M_\gamma = [M_\gamma] + F_\gamma = N_\gamma + F_\gamma \qquad M_\delta = [M_\delta] + F_\delta = N_\delta + F_\delta. \tag{5.4}$$

Case 1 is said to be at hand if M_γ and M_δ both are integers. Then (5.3) takes the form;

$$N_\gamma + N_\delta = N, \quad \gamma \cdot N_\gamma + \delta \cdot N_\delta = N. \tag{5.5}$$

Case 2 is said to be at hand if M_γ and M_δ not both are integers. Then none of them is an integer, since N_γ and N_δ add to an integer. Moreover, as is readily checked;

$$F_\gamma + F_\delta = 1, \quad N_\gamma + N_\delta = N - 1, \quad N - (\gamma \cdot N_\gamma + \delta \cdot N_\delta) = \gamma \cdot F_\gamma + \delta \cdot F_\delta. \tag{5.6}$$

The special patterns are first specified, and then is shown that they satisfy (4.1) and (5.1).

The boundary pattern $s(N;\gamma;\delta;\mathbf{b})$

It is presumed that N, γ and δ are such that $N_\gamma \ge 1$ and $N_\delta \ge 1$. For this size pattern N_γ units are given the s - value γ, and N_δ units the value δ. In Case 1 all units thereby get s - values. In Case 2 one unit remains, which is given the s - value;

$$s_N = N - (\gamma \cdot N_\gamma + \delta \cdot N_\delta), \text{ which by (5.2) also means } s_N = \gamma \cdot F_\gamma + \delta \cdot F_\delta. \tag{5.7}$$

The middle pattern $s(N;\gamma;\delta;\mathbf{m})$

It is presumed that N, γ and δ are such that $\gamma \le 1 - (\gamma + \delta - 2)/(N - 2) \le \delta$. For this size pattern s-values are assigned as follows. The size values γ and δ re given to one unit each. All remaining units are given the s-value;

$$s = 1-(\gamma+\delta-2)/(N-2). \tag{5.8}$$

The even spread pattern $s(N;\gamma;\delta;e)$

It is presumed that N, γ and δ are such that $N_\gamma \geq 2$ and $N_\delta \geq 2$. The s-values are allocated as follows.

$$\text{If } (1-\gamma)\cdot N_\gamma < (\delta-1)\cdot N_\delta; \tag{5.9}$$

$$s_i = \gamma+(i-1)\cdot(1-\gamma)/(N_\gamma-1), \quad i=1,2,\ldots,N_\gamma, \tag{5.10}$$

$$s_i = \delta-(N_\delta+N_\gamma-i)\cdot(1-\gamma)\cdot N_\gamma/[N_\delta\cdot(N_\delta-1)], \quad i=N_\gamma+1, N_\gamma+2,\ldots,N_\gamma+N_\delta. \tag{5.11}$$

$$\text{If } (1-\gamma)\cdot N_\gamma \geq (\delta-1)\cdot N_\delta; \tag{5.12}$$

$$s_i = \gamma+(i-1)\cdot(\delta-1)\cdot N_\delta/[N_\gamma\cdot(N_\gamma-1)], \quad i=1,2,\ldots,N_\gamma, \tag{5.13}$$

$$s_i = \delta-(N_\delta+N_\gamma-i)\cdot(\delta-1)/(N_\delta-1), \quad i=N_\gamma+1, N_\gamma+2,\ldots,N_\gamma+N_\delta. \tag{5.14}$$

In Case 1 all units get s-values by (5.9)-(5.11). In Case 2 one unit remains, which is given the s-value in (5.7).

Thereby the size patterns are defined and the remaining task is to show that they satisfy (4.1) and (5.1). For the middle pattern the verifications are straightforward. The same holds for the boundary pattern, when noted that s_N in (5.7) is a linear combination of γ and δ and, hence, lies in the interval $[\gamma,\delta]$. For the even pattern we start with case (5.9). Formulas (5.10) and (5.11) readily yield;

$$\sum_{i=1}^{N_\gamma} s_i = \gamma\cdot N_\gamma + N_\gamma\cdot(1-\gamma)/2 \quad \text{and} \quad \sum_{i=N_\gamma+1}^{N_\gamma+N_\delta} s_i = \delta\cdot N_\delta - N_\gamma\cdot(1-\gamma)/2, \tag{5.15}$$

which in turn yields;

$$\sum_{i=1}^{N_\gamma+N_\delta} s_i = \gamma\cdot N_\gamma + \delta\cdot N_\delta. \tag{5.16}$$

In Case 1 relation (5.1) follows from (5.16) and (5.5). In Case 2 it follows from the definition (5.7) of s_N. It remains to show (5.1). The s-values in (5.11) are generated as;

$$s_i = \delta-(N_\delta+N_\gamma-i)\cdot(\delta-\varepsilon)/(N_\delta-1), \quad i=N_\gamma+1, N_\gamma+2,\ldots,N_\gamma+N_\delta, \tag{5.17}$$

where ε solves the equality version $(1-\gamma)\cdot N_\gamma = (\delta-\varepsilon)\cdot N_\delta$, of the inequality in (5.9);

$$\varepsilon = \delta-(1-\gamma)\cdot N_\gamma/N_\delta. \tag{5.18}$$

(5.9) yields that $1 \leq \varepsilon$, and (5.18) that $\varepsilon \leq \delta$. Hence, all s-values in (5.11) lie in $[\gamma,\delta]$. The same holds for the s-values in (5.10). That s_N in (5.7) satisfies (5.1) follows from (5.7). Then case (5.12) can be treated quite analogously, and is left to the reader. This concludes the proof.

6 On the magnitude of estimator bias

6.1 Factors that affect the bias

6.1.1 Introduction

For a practitioner who considers to use Pareto πps, a crucial question is;

Will the estimator bias be negligible in my particular sampling-estimation situation? (6.1)

Answers to (6.1) in terms of practically available parameters are given in next section. They are with necessity a bit complex, since the approximation accuracy depends on several factors, notably those listed below and commented on thereafter.

- The study variable.
- The tolerance limit for "negligibility".
- The size value pattern.
- The population size.
- The sample size.

As regards the study variable, we confine to the case with non-negative variable, which is the typical situation in practice. Hence, subsequent statements about bias shall be interpreted in according to (3.6). It is left to the reader to make appropriate modifications for situations with sign changing study variable.

6.1.2 Tolerance level for negligibility

There is of course no single answer to how large at most a "negligible" bias may be. It depends on the intended use of the statistic and the magnitude of other survey errors, notably the sampling error. We believe that most statisticians mean that 1%, or even 2%, relative bias is negligible, a reason being that the sampling error commonly is a good deal larger.

6.1.3 Dependence on the size value variation

As said in Section 4.1, for an approximation problem to be at hand the size values must exhibit variation, having the aspects ***range*** and ***shape***. The range is the interval $[s_{min}, s_{max}] = [\gamma, \delta]$. For a fixed size pattern shape Ψ-values increase with range. Based on experience we believe that in practical surveys $s_{max} = \delta$ seldom is larger than 5 and $s_{min} = \gamma$ seldom is smaller than 0.1. Some motivation is given below.

The surveyor disposes of the size values, in the sense that "preliminary" values may be modified. If the frame comprises units with very small (preliminary) size values, such units are often either definition-wise excluded from the target population or given larger size values.

If size values vary very much over the entire population, there are often grounds for stratification by size before sampling, followed by selection of independent samples from the strata. (A typical example is provided by an enterprise survey with number of employees as size. Then it is often natural to divide into strata of types "very big", "big" and "small" enterprises. Mostly the "very big stratum" is totally inspected.) The strata then take population roles, and s_{max} and s_{min} in strata are considerably smaller/larger than in the entire population.

As regards size pattern shape our experience says, as stated in Section 4.2.7, that the boundary pattern, which is most adverse for good approximation, is very unusual in practice.

Table 6.1 below introduces, for later use, a broad categorization of size value patterns.

Table 6.1	**Some categories of size value patterns**			
	Category A	**Category B**	**Category C**	**Category D**
Size pattern shape	In the vicinity of even spread.	In the vicinity of even spread.	No restriction.	No restriction.
s_{max}	≤ 5	≤ 10	≤ 5	≤ 10
s_{min}	≥ 0.1	≥ 0.05	≥ 0.1	≥ 0.05
Comment	Most practical sampling situations are believed to fall in this category.	"Normal" pattern shape, while s_{max} and/or s_{min} may be extreme	"Normal" s_{max} and s_{min}, while pattern shape may be extreme (e.g. of boundary type).	Pattern shape as well as s_{max} and/or s_{min} may be extreme.

6.1.4 Dependence on population and sample sizes

As discussed in Section 4.2.6, approximation accuracy improves as population size increases (while s_{min}, s_{max}, α and n are fixed). Some of the figures in Table 6.2 below are based on extrapolation from available numerical data by employing the mentioned principle.

As regards dependence on sample size we refer to Sections 4.4 and 4.2.5.

6.2 Conditions for negligible estimator bias

The full numerical material derived to provide answers to questions (3.8) and (3.9) is presented in Appendices 1 and 2. It is somewhat difficult, though, to overview it as it stands in the appendices. The following Table 6.2 summarizes the numerical findings at the prize of some coarsening. In some cases it may be helpful to consult the more detailed material in the appendices. Population sizes smaller than 25 were not considered in the study.

Table 6.2. Sample sizes that imply negligible bias

N = population size. α states that the sample size should not exceed $n_{m,\alpha}$ in (4.4). n_0 specifies a sufficiently large sample size, under the α-restriction, for negligible bias with specified tolerance. The study variable is presumed to be non-negative. Values for categories A and B are based on the weak quasi envelope (4.18), those for categories C and D on the (general) quasi envelope (4.7).

Size Pattern category See Table 6.1	Tolerance limit for negligibility								
	2%			1%			0.5%		
	N	α	n_0	N	α	n_0	N	α	n_0
A	≥ 25	1	1	≥40	1	1	≥80	1	1
				[25,40)	1	3	[50,80)	1	3
							[40,50)	1	4
B	≥80	1	1	≥80	1	1	≥ 125	1	1
	[25,80)	1	2	[40,80)	1	3	[100,125)	1	3
							[80,100)	1	4
C	≥ 100	1	1	≥ 125	1	1	≥ 175	1	1
	≥ 80	0.9	1	≥ 100	0.9	1	≥ 150	0.9	1
	≥ 40	0.8	1	≥ 80	0.8	1	≥ 100	0.8	1
	≥ 25	0.5	1	≥ 40	0.5	1	≥ 80	0.5	1
D	≥ 150	0.9	1	≥ 175	0.8	1	≥ 125	0.5	1
	≥ 125	0.8	1	≥ 80	0.5	1	[100,125)	0.5	3
	≥ 80	0.5	1						
	[50,80)	0.5	2						

From Remarks 3.1 and 4.1 follows that the sufficient sample sizes in Table 6.2 are more or less conservative and, hence, "overly safe". In particular, one should not conclude that the bias necessarily is larger than "guaranteed" for sample sizes that are smaller than stated n_0-values.

Our ***overall conclusion*** from the findings is as follows. Although the figures in Table 6.1 are conservative, we believe that they in most practical situations lead to the conclusion that the bias is negligible for all admissible sample sizes and, hence, that Pareto πps can safely be employed.

References

Aires, N. (1999). Algorithms to Find Exact Inclusion Probabilities for Conditional Poisson Sampling and Pareto πps Sampling Designs. *Methodology and Computing in Applied Probability*, **4**, 459-473.

Aires, N. (2000). Comparison between Conditional Poisson Sampling and Pareto πps Sampling Designs. *J of Statist Planning and Inference*, **82**, No. 2.

Rosén, B. (1997). On Sampling with Probability Proportional to Size. *J of Statist Planning and Inference*, **62**, 159-191.

Rosén, B. (2000). On Inclusion Probabilities for Order Sampling. To appear in *J of Statist Planning and Inference.*

Saavedra (1995). Fixed Sample Size PPS Approximations with a Permanent Random Number. *1995 Joint Statistical Meetings American Statistical Association.* Orlando, Florida.

Appendices with numerical results

The following Appendices 1 and 2 contain numerical results. The basic quantities in this context are Ψ - sequence values for the three particular size value patterns (even spread, middle and boundary) for a variety of values for the parameters N, s_{min} and s_{max}. The complete collection of basic data is, modulo possible visual difficulties, given by the Ψ - sequence graphs in Appendix 2. The corresponding numerical values constituted basis for deriving numerical values for quantities of more direct practical relevance, upper sequences and sufficient sample sizes, which are presented in Appendix 1. When pondering upon these numbers, it is often illuminating to look at the corresponding graphs in Appendix 2.

Appendix 1. Upper sequences and sufficient sample sizes

This appendix presents values for ***upper sequences***; general, for α-admissible sample sizes, in weak version (see (4.11) and (4.13)), and ***sufficient sample sizes***; general, in (β, α) - version, in weak version (see (4.15) and (4.16)). For space reasons presentation of upper sequences are confined to their first 5 items. Considered parameter values are listed below.

Tables 1-4: Upper sequences for $\mathbf{s_{max} = 2}$, $s_{min} = 0.5, 0.2, 0.1, 0.05$, $N = 25, 40, 50, 80, 100$, $\alpha = 1, 0.9, 0.8, 0.5$.

Tables 5-8: β-and (β, α)-sufficient sample sizes for $\mathbf{s_{max} = 2}$, $s_{min} = 0.5, 0.2, 0.1, 0.05$, $N = 25, 40, 50, 80, 100$, $\beta = 5, 2, 1, 0.5\%$, $\alpha = 1, 0.9, 0.8, 0.5$.

Tables 9-12: Upper sequences for $\mathbf{s_{max} = 4}$, $s_{min} = 0.5, 0.2, 0.1, 0.05$, $N = 25, 40, 50, 80, 100$, $\alpha = 1, 0.9, 0.8, 0.5$.

Tables 13-16: β-and (β, α)-sufficient sample sizes for $\mathbf{s_{max} = 4}$, $s_{min} = 0.5, 0.2, 0.1, 0.05$, $N = 25, 40, 50, 80, 100$, $\beta = 5, 2, 1, 0.5\%$, $\alpha = 1, 0.9, 0.8, 0.5$.

Tables 17-20: Upper sequences for $\mathbf{s_{max} = 5}$, $s_{min} = 0.5, 0.2, 0.1, 0.05$, $N = 25, 40, 50, 80, 100, 125, 150, 175, 200$ (for $s_{min} = 0.5$ only for N up to 100), $\alpha = 1, 0.9, 0.8, 0.5$.

Tables 21-24: β-and (β, α)-sufficient sample sizes for $\mathbf{s_{max} = 5}$, $s_{min} = 0.5, 0.2, 0.1, 0.05$, $N = 25, 40, 50, 80, 100, 125, 150, 200$ (for $s_{min} = 0.5$ only for N up to 100), $\beta = 5, 2, 1, 0.5\%$, $\alpha = 1, 0.9, 0.8, 0.5$.

Tables 25-28: Upper sequences for $\mathbf{s_{max} = 10}$, $s_{min} = 0.5, 0.2, 0.1, 0.05$, $N = 25, 40, 50, 80, 100, 125, 150, 175, 200$ (for $s_{min} = 0.5$, only for N up to 100), $\alpha = 1, 0.9, 0.8, 0.5$.

Tables 29-32: β-and (β, α)-sufficient sample sizes for $\mathbf{s_{max} = 10}$, $s_{min} = 0.5, 0.2, 0.1, 0.05$, $N = 25, 40, 50, 80, 100, 125, 150, 200$ (for $s_{min} = 0.5$ only for N up to 100), $\beta = 5, 2, 1, 0.5\%$, $\alpha = 1, 0.9, 0.8, 0.5$.

Tables 33-34: Weak upper sequences for $\mathbf{s_{max} = 2}$ and **4**, $s_{min} = 0.5, 0.2, 0.1, 0.05$, $N = 25, 40, 50, 80, 100$.

Tables 35-36: Weakly β-sufficient sample sizes for s_{max} = **2** and **4**, s_{min}= 0.5, 0.2, 0.1, 0.05, N = 25, 40, 50, 80, 100, β = 5, 2, 1, 0.5%.

Tables 37-38: Weak upper sequences for s_{max} = **5** and **10**, s_{min} = 0.5, 0.2, 0.1, 0.05, N = 25, 40, 50, 80, 100, 125, 150, 175, 200 (for s_{min}=0.5 only for N up to 100).

Tables 39-40: Weakly β-sufficient sample sizes for s_{max} = **5** and **10**, s_{min} = 0.5, 0.2, 0.1, 0.05, N = 25, 40, 50, 80, 100, 125, 150, 200 (for s_{min}=0.5 only for N up to 100), β = 5, 2, 1, 0.5%.

Tables for s_{max} = 2

Table 1. Values (in %) for $\Gamma(n_0; N; s_{min}; s_{max})$ in (4.11).

			s_{max} = 2. s_{min} =			
N	**n_0**	**n_m**	**0.5**	**0.2**	**0.1**	**0.05**
25	**1-5**	**12**	0.28	1.7	8.3	12.9
40	**1-5**	**19**	0.11	0.72	3.4	3.9
50	**1-5**	**24**	0.07	0.44	1.9	3.7
80	**1-5**	**39**	0.03	0.20	0.77	3.4
100	**1-5**	**49**	0.02	0.13	0.45	2.0

Table 2. Values (in %) for $\Gamma_{0.9}(n_0; N; s_{min}; s_{max})$ in (4.13).

			s_{max} = 2. s_{min} =			
N	**n_0**	**$n_{m,0.9}$**	**0.5**	**0.2**	**0.1**	**0.05**
25	**1-5**	**11**	0.28	1.7	2.8	3.7
40	**1-5**	**18**	0.11	0.72	1.7	1.6
50	**1-5**	**22**	0.07	0.41	0.77	0.83
80	**1-5**	**36**	0.03	0.17	0.43	0.54
100	**1-5**	**45**	0.02	0.11	0.26	0.32

Table 3. Values (in %) for $\Gamma_{0.8}(n_0; N; s_{min}; s_{max})$ in (4.13).

			s_{max} = 2. s_{min} =			
N	**n_0**	**$n_{m,0.8}$**	**0.5**	**0.2**	**0.1**	**0.05**
25	**1-5**	**10**	0.28	1.1	1.3	1.8
40	**1-5**	**16**	0.11	0.44	0.59	0.46
50	**1-5**	**20**	0.07	0.29	0.36	0.40
80	**1-5**	**32**	0.03	0.11	0.15	0.13
100	**1-5**	**40**	0.02	0.07	0.10	0.08

Table 4. Values (in %) for $\Gamma_{0.5}(n_0; N; s_{min}; s_{max})$ in (4.13).

			s_{max} = 2. s_{min} =			
N	**n_0**	**$n_{m,0.5}$**	**0.5**	**0.2**	**0.1**	**0.05**
25	**1-5**	**6**	0.28	0.37	0.27	0.35
40	**1-5**	**10**	0.11	0.13	0.11	0.14
50	**1-5**	**12**	0.07	0.09	0.07	0.10
80	**1-5**	**20**	0.03	0.03	0.03	0.03
100	**1-5**	**25**	0.02	0.02	0.03	0.02

Table 5. β-sufficient sample sizes.

			s_{max} = 2. s_{min} =			
N	**n_m**	**β**	**0.5**	**0.2**	**0.1**	**0.05**
25	**12**	**5%**	1	1	none	none
		2%	1	1	none	none
		1-0.5%	1	none	none	none
40	**19**	**5%**	1	1	1	1
		2%	1	1	none	none
		1%	1	1	none	none
		0.5%	1	none	none	none
50	**24**	**5%**	1	1	1	1
		2%	1	1	1	none
		1%	1	1	none	none
		0.5%	1	1	none	none
80	**39**	**5%**	1	1	1	1
		2-1%	1	1	1	none
		0.5%	1	1	none	none
100	**49**	**5%**	1	1	1	1
		2-0.5%	1	1	1	none

Table 6. (β, 0.9)-sufficient sample sizes.

			s_{max} = 2. s_{min} =			
N	**$n_{m,0.9}$**	**β**	**0.5**	**0.2**	**0.1**	**0.05**
25	**11**	**5%**	1	1	1	1
		2%	1	1	none	none
		1-0.5%	1	none	none	none
40	**18**	**5%**	1	1	1	1
		2%	1	1	1	1
		1%	1	1	none	none
		0.5%	1	none	none	none
50	**22**	**5%**	1	1	1	1
		2%	1	1	1	1
		1%	1	1	1	1
		0.5%	1	1	none	none
80	**36**	**5%**	1	1	1	1
		2-1%	1	1	1	1
		0.5%	1	1	1	none
100	**45**	**5%**	1	1	1	1
		2-0.5%	1	1	1	1

Table 7. $(\beta, 0.8)$-sufficient sample sizes.						
			$s_{max} = 2.$ $s_{min} =$			
N	n_m	β	0.5	0.2	0.1	0.05
25	10	5-2%	1	1	1	1
		1-0.5%	1	none	none	none
40	16	5-1%	1	1	1	1
		0.5%	1	none	none	1
50	20	5-0.5%	1	1	1	1
80	32	5-0.5%	1	1	1	1
100	40	5-0.5%	1	1	1	1

Table 8. $(\beta, 0.5)$-sufficient sample sizes.						
			$s_{max} = 2.$ $s_{min} =$			
N	$n_{m,0.5}$	β	0.5	0.2	0.1	0.05
25	6	5-2%	1	1	1	1
		1-0.5%	1	1	1	1
40	10	5-1%	1	1	1	1
		0.5%	1	1	1	1
50	12	5-0.5%	1	1	1	1
80	20	5-0.5%	1	1	1	1
100	25	5-0.5%	1	1	1	1

Tables for $s_{max} = 4$

Table 9. Values (in %) for $\Gamma(n_0; N; s_{min}; s_{max})$ in (4.11).						
			$s_{max} = \delta = 4.$ $s_{min} =$			
N	n_0	n_m	0.5	0.2	0.1	0.05
25	1	6	1.3	3.0	13.2	30.9
	2		1.0	3.0	13.2	30.9
	3		0.71	3.0	13.2	30.9
	4		0.68	3.0	13.2	30.9
	5		0.66	3.0	13.2	30.9
40	1	9	0.58	1.3	4.0	4.4
	2		0.50	1.3	4.0	4.4
	3		0.41	1.3	4.0	4.4
	4		0.33	1.3	4.0	4.4
	5		0.30	1.3	4.0	4.4
50	1	12	0.39	0.90	3.1	13.5
	2		0.35	0.90	3.1	13.5
	3		0.30	0.90	3.1	13.5
	4		0.26	0.90	3.1	13.5
	5		0.21	0.90	3.1	13.5
80	1	19	0.16	0.40	1.2	3.7
	2		0.15	0.40	1.2	3.7
	3		0.14	0.40	1.2	3.7
	4		0.13	0.40	1.2	3.7
	5		0.12	0.40	1.2	3.7
100	1	24	0.11	0.26	0.91	3.9
	2-3		0.10	0.26	0.91	3.9
	4		0.09	0.26	0.91	3.9
	5		0.08	0.26	0.91	3.9

Table 10. Values (in %) for $\Gamma_{0.9}(n_0; N; s_{min}; s_{max})$ in (4.13).						
			$s_{max} = 4.$ $s_{min} =$			
N	n_0	$n_{m,0.5}$	0.5	0.2	0.1	0.05
25	1	5	1.3	3.0	3.9	4.8
	2		1.0	3.0	3.9	4.8
	3		0.71	3.0	3.9	4.8
	4		0.68	3.0	3.9	4.8
	5		0.52	3.0	3.9	4.8
40	1	9	0.58	1.3	4.0	4.4
	2		0.50	1.3	4.0	4.4
	3		0.41	1.3	4.0	4.4
	4		0.33	1.3	4.0	4.4
	5		0.30	1.3	4.0	4.4
50	1	11	0.39	0.90	2.1	3.6
	2		0.35	0.90	2.1	3.6
	3		0.30	0.90	2.1	3.6
	4		0.26	0.90	2.1	3.6
	5		0.21	0.90	2.1	3.6
80	1	18	0.16	0.40	1.0	1.7
	2		0.15	0.40	1.0	1.7
	3		0.14	0.40	1.0	1.7
	4		0.13	0.40	1.0	1.7
	5		0.12	0.40	1.0	1.7
100	1	22	0.11	0.26	0.70	1.0
	2-3		0.10	0.26	0.70	1.0
	4		0.09	0.26	0.70	1.0
	5		0.08	0.26	0.70	1.0

Table 11. Values (in %) for $\Gamma_{0.8}(n_0; N; s_{min}; s_{max})$ in (4.13).						
			$s_{max} = 4.$ $s_{min} =$			
N	n_0	$n_{m,0.8}$	0.5	0.2	0.1	0.05
25	1	5	1.33	3.0	3.9	4.8
	2		1.01	3.0	3.9	4.8
	3		0.71	3.0	3.9	4.8
	4		0.68	3.0	3.9	4.8
	5		0.52	3.0	3.9	4.8
40	1	8	0.58	1.2	2.0	1.6
	2		0.50	1.2	2.0	1.6
	3		0.41	1.2	2.0	1.6
	4		0.33	1.2	2.0	1.6
	5		0.30	1.2	2.0	1.6

Table 12. Values (in %) for $\Gamma_{0.5}(n_0; N; s_{min}; s_{max})$ in (4.13).						
			$s_{max} = 4.$ $s_{min} =$			
N	n_0	$n_{m,0.5}$	0.5	0.2	0.1	0.05
25	1	3	1.3	1.3	1.32	1.32
	2		1.0	1.3	1.00	1.00
	3		0.71	1.3	0.86	0.72
40	1	5	0.58	0.58	0.64	0.58
	2		0.50	0.50	0.64	0.50
	3		0.41	0.48	0.64	0.41
	4		0.33	0.48	0.64	0.39
	5		0.30	0.48	0.64	0.39

50	1	10	0.39	0.78	1.2	1.5
	2		0.35	0.78	1.2	1.5
	3		0.30	0.78	1.2	1.5
	4		0.26	0.78	1.2	1.5
	5		0.21	0.78	1.2	1.5
80	1	16	0.16	0.34	0.50	0.70
	2		0.15	0.34	0.50	0.70
	3		0.14	0.34	0.50	0.70
	4		0.13	0.34	0.50	0.70
	5		0.12	0.34	0.50	0.70
100	1	20	0.11	0.23	0.40	0.43
	2		0.10	0.23	0.40	0.43
	3		0.10	0.23	0.40	0.43
	4		0.09	0.23	0.40	0.43
	5		0.08	0.23	0.40	0.43

50	1	6	0.39	0.39	0.39	0.39
	2		0.35	0.34	0.34	0.34
	3		0.30	0.30	0.30	0.30
	4		0.26	0.29	0.28	0.26
	5		0.21	0.29	0.28	0.21
80	1	10	0.16	0.16	0.16	0.16
	2		0.15	0.15	0.15	0.16
	3		0.14	0.14	0.14	0.16
	4		0.13	0.13	0.14	0.16
	5		0.12	0.13	0.14	0.16
100	1	12	0.11	0.11	0.11	0.10
	2		0.10	0.10	0.10	0.10
	3		0.10	0.10	0.10	0.10
	4		0.09	0.09	0.09	0.09
	5		0.08	0.09	0.08	0.08

Table 13. β - sufficient sample sizes.

N	n_m	β	$s_{max}=4$. $s_{min}=$ 0.5	0.2	0.1	0.05
25	6	5%	1	1	none	none
		2%	1	6	none	none
		1%	3	none	none	none
		0.5%	none	none	none	none
40	9	5%	1	1	1	1
		2%	1	1	none	none
		1%	1	none	none	none
		0.5%	2	none	none	none
50	12	5%	1	1	1	none
		2%	1	1	1	none
		1%	1	1	1	none
		0.5%	1	none	1	none
80	19	5%	1	1	1	1
		2%	1	1	1	none
		1 - 0.5%	1	1	none	none
100	25	5%	1	1	1	1
		2%	1	1	1	none
		1%	1	1	1	none
		0.5%	1	1	none	none

Table 14. (β, 0.9) - sufficient sample sizes.

N	$n_{m,0.9}$	β	$s_{max}=4$. $s_{min}=$ 0.5	0.2	0.1	0.05
25	5	5%	1	1	1	1
		2%	1	none	none	none
		1%	3	none	none	none
		0.5%	none	none	none	none
40	9	5%	1	1	1	1
		2%	1	1	none	none
		1%	1	none	none	none
		0.5%	2	none	none	none
50	11	5%	1	1	1	1
		2%	1	1	1	none
		1%	1	1	1	none
		0.5%	1	none	1	none
80	18	5%	1	1	1	1
		2%	1	1	1	1
		1 - 0.5%	1	1	none	none
100	22	5%	1	1	1	1
		2%	1	1	1	1
		1%	1	1	1	none
		0.5%	1	1	none	none

Table 15. (β, 0.8) - sufficient sample sizes.

N	$n_{m,0.8}$	β	$s_{max}=4$. $s_{min}=$ 0.5	0.2	0.1	0.05
25	5	5%	1	1	1	1
		2%	1	none	none	none
		1%	3	none	none	none
		0.5%	none	none	none	none
40	8	5 - 2%	1	1	1	1
		1%	1	none	none	none
		0.5%	2	none	none	none
50	10	5 - 2%	1	1	1	1
		1%	1	1	1	none
		0.5%	1	none	1	none
80	16	5 - 1%	1	1	1	1
		0.5%	1	1	none	none
100	20	5 - 0.5%	1	1	1	1

Table 16. (β, 0.5) - sufficient sample sizes.

N	$n_{m,0.5}$	β	$s_{max}=4$. $s_{min}=$ 0.5	0.2	0.1	0.05
25	3	5%	1	1	1	1
		2%	1	1	1	1
		1%	3	none	3	3
		0.5%	none	none	none	none
40	5	5 - 2%	1	1	1	1
		1%	1	1	1	1
		0.5%	2	2	none	2
50	6	5 - 2%	1	1	1	1
		1%	1	1	1	1
		0.5%	1	1	1	1
80	10	5 - 1%	1	1	1	1
		0.5%	1	1	1	1
100	12	5 - 0.5%	1	1	1	1

Tables for s_{max} = 5

Table 17. Values (in %) for $\Gamma(n_0; N; s_{min}; s_{max})$ in (4.11).

N	n_0	n_m	s_{max} =5. s_{min} =			
			0.5	0.2	0.1	0.05
25	1	4	2.0	4.7	4.9	5.0
	2		1.4	4.7	4.9	5.0
	3		1.1	4.7	4.9	5.0
	4		0.86	4.7	4.9	5.0
40	1	7	0.91	2.1	4.2	4.6
	2		0.73	2.1	4.2	4.6
	3		0.56	2.1	4.2	4.6
	4		0.43	2.1	4.2	4.6
	5		0.39	2.1	4.2	4.6
50	1	9	0.61	1.1	4.1	4.3
	2		0.52	1.1	4.1	4.3
	3		0.43	1.1	4.1	4.3
	4		0.34	1.1	4.1	4.3
	5		0.28	1.1	4.1	4.3
80	1	15	0.26	0.50	2.0	3.8
	2		0.24	0.50	2.0	3.8
	3		0.21	0.50	2.0	3.8
	4		0.19	0.50	2.0	3.8
	5		0.16	0.50	2.0	3.8
100	1	19	0.17	0.34	1.1	3.8
	2		0.16	0.34	1.1	3.8
	3		0.15	0.34	1.1	3.8
	4		0.13	0.34	1.1	3.8
	5		0.12	0.34	1.1	3.8
125	1 - 5	24		0.23	0.88	3.7
150	1 - 5	29		0.17	0.55	2.4
175	1 - 5	34		0.12	0.42	1.6
200	1 - 5	39		0.09	0.34	1.1

Table 18. Values (in %) for $\Gamma_{0.9}(n_0; N; s_{min}; s_{max})$ in (4.13).

N	n_0	$n_{m,0.9}$	s_{max} =5. s_{min} =			
			0.5	0.2	0.1	0.05
25	1	4	2.0	4.7	4.9	5.0
	2		1.4	4.7	4.9	5.0
	3		1.1	4.7	4.9	5.0
	4		0.86	4.7	4.9	5.0
40	1	7	0.91	2.1	4.2	4.6
	2		0.73	2.1	4.2	4.6
	3		0.56	2.1	4.2	4.6
	4		0.43	2.1	4.2	4.6
	5		0.39	2.1	4.2	4.6
50	1	9	0.61	1.1	4.1	4.3
	2		0.52	1.1	4.1	4.3
	3		0.43	1.1	4.1	4.3
	4		0.34	1.1	4.1	4.3
	5		0.28	1.1	4.1	4.3
80	1	14	0.26	0.50	1.3	1.6
	2		0.24	0.50	1.3	1.6
	3		0.21	0.50	1.3	1.6
	4		0.19	0.50	1.3	1.6
	5		0.16	0.50	1.3	1.6
100	1	18	0.17	0.34	0.99	1.8
	2		0.16	0.34	0.99	1.8
	3		0.15	0.34	0.99	1.8
	4		0.13	0.34	0.99	1.8
	5		0.12	0.34	0.99	1.8
125	1 - 5	22		0.23	0.66	0.94
150	1 - 5	27		0.17	0.47	0.80
175	1 - 5	31		0.12	0.33	0.47
200	1 - 5	36		0.09	0.28	0.45

Table 19. Values (in %) for $\Gamma_{0.8}(n_0; N; s_{min}; s_{max})$ in (4.13).

N	n_0	$n_{m,0.8}$	s_{max} =5. s_{min} =			
			0.5	0.2	0.1	0.05
25	1	4	2.0	4.7	4.9	5.0
	2		1.4	4.7	4.9	5.0
	3		1.11	4.7	4.9	5.0
	4		0.86	4.7	4.9	5.0
40	1	6	0.91	1.5	1.9	1.6
	2		0.73	1.5	1.9	1.6
	3		0.56	1.5	1.9	1.6
	4		0.43	1.5	1.9	1.6
	5		0.39	1.5	1.9	1.6
50	1	8	0.61	1.1	2.1	1.6
	2		0.52	1.1	2.1	1.6
	3		0.43	1.1	2.1	1.6
	4		0.34	1.1	2.1	1.6
	5		0.28	1.1	2.1	1.6
80	1	12	0.26	0.41	0.55	0.65
	2		0.24	0.41	0.55	0.65
	3		0.21	0.41	0.55	0.65
	4		0.19	0.41	0.55	0.65
	5		0.16	0.41	0.55	0.65

Table 20. Values (in %) for $\Gamma_{0.5}(n_0; N; s_{min}; s_{max})$ in (4.13).

N	n_0	$n_{m,0.5}$	s_{max} =5. s_{min} =			
			0.5	0.2	0.1	0.05
25	1	2	2.0	2.0	2.0	2.0
	2		1.4	1.8	1.4	1.3
40	1	4	0.91	0.91	0.92	0.91
	2		0.73	0.73	0.92	0.73
	3		0.56	0.56	0.92	0.56
	4		0.43	0.40	0.92	0.50
50	1	5	0.61	0.61	0.64	0.61
	2		0.52	0.52	0.64	0.52
	3		0.43	0.52	0.64	0.43
	4		0.34	0.52	0.64	0.42
	5		0.28	0.52	0.64	0.42
80	1	8	0.26	0.26	0.26	0.26
	2		0.24	0.24	0.24	0.25
	3		0.21	0.21	0.21	0.25
	4		0.19	0.21	0.19	0.25
	5		0.16	0.21	0.16	0.25

100	1	16	0.17	0.31	0.50	0.72
	2		0.16	0.31	0.50	0.72
	3		0.15	0.31	0.50	0.72
	4		0.13	0.31	0.50	0.72
	5		0.12	0.31	0.50	0.72
125	1-2	20		0.21	0.36	0.36
	3			0.21	0.36	0.36
	4-5			0.21	0.36	0.36
150	1-2	24		0.15	0.23	0.24
	3-4			0.15	0.23	0.24
	5			0.15	0.23	0.24
175	1-3	28		0.11	0.20	0.17
	4-5			0.11	0.20	0.17
200	1-3	32		0.08	0.14	0.14
	4-5			0.08	0.14	0.14

Table 21. β **- sufficient sample sizes.**

N	n_m	β	$s_{max}=5.$ $s_{min}=$ 0.5	0.2	0.1	0.05
25	4	5%	1	1	1	1
		2%	2	none	none	none
		1%	4	none	none	none
		0.5%	none	none	none	none
40	7	5%	1	1	1	1
		2-1%	1	none	none	none
		0.5%	4	none	none	none
50	9	5%	1	1	1	1
		2%	1	1	none	none
		1%	1	none	none	none
		0.5%	3	none	none	none
80	15	5%	1	1	1	1
		2%	1	1	none	none
		1 - 0.5%	1	1	none	none
100	19	5%	1	1	1	1
		2%	1	1	1	none
		1%	1	1	none	none
		0.5%	1	1	none	none
125	24	5%		1	1	1
		2-1%		1	1	none
		0.5%		1	none	none
150	29	5%		1	1	1
		2-1%		1	1	none
		0.5%		1	29	none
175	34	5-2%		1	1	1
		1-0.5%		1	1	none
200	39	5-2%		1	1	1
		1-0.5%		1	1	none

Table 23. $(\beta, 0.8)$ **- sufficient sample sizes.**

N	$n_{m,0.8}$	β	$s_{max}=5.$ $s_{min}=$ 0.5	0.2	0.1	0.05
25	4	5%	1	1	1	1
		2%	2	none	none	none
		1%	4	none	none	none
		0.5%	none	none	none	none

100	1	10	0.17	0.17	0.17	0.17
	2		0.16	0.16	0.16	0.16
	3		0.15	0.15	0.16	0.16
	4		0.13	0.13	0.16	0.16
	5		0.12	0.13	0.16	0.16
125	1-2	12		0.11	0.11	0.11
	3			0.10	0.10	0.10
	4			0.09	0.09	0.09
150	1-2	15		0.08	0.08	0.08
	3-4			0.07	0.07	0.07
	5			0.06	0.06	0.06
175	1-3	17		0.06	0.06	0.06
	4-5			0.05	0.05	0.05
200	1-3	20		0.04	0.04	0.04
	4-5			0.04	0.04	0.04

Table 22. $(\beta, 0.9)$ **- sufficient sample sizes.**

N	$n_{m,0.9}$	β	$s_{max}=5.$ $s_{min}=$ 0.5	0.2	0.1	0.05
25	4	5%	1	1	1	1
		2%	2	none	none	none
		1%	4	none	none	none
		0.5%	none	none	none	none
40	7	5%	1	1	1	1
		2-1%	1	none	none	none
		0.5%	4	none	none	none
50	9	5%	1	1	1	1
		2%	1	1	none	none
		1%	1	none	none	none
		0.5%	3	none	none	none
80	14	5%	1	1	1	1
		2%	1	1	1	1
		1 - 0.5%	1	1	none	none
100	18	5%	1	1	1	1
		2%	1	1	1	1
		1%	1	1	1	none
		0.5%	1	1	none	none
125	22	5%		1	1	1
		2-1%		1	1	1
		0.5%		1	none	none
150	27	5%		1	1	1
		2-1%		1	1	1
		0.5%		1	1	none
175	31	5%		1	1	1
		1-0.5%		1	1	1
200	36	5-2%		1	1	1
		1-0.5%		1	1	1

Table 24. $(\beta, 0.5)$ **- sufficient sample sizes.**

N	$n_{m,0.5}$	β	$s_{max}=5.$ $s_{min}=$ 0.5	0.2	0.1	0.05
25	2	5%	1	1	1	1
		2%	1	1	1	1
		1%	none	none	none	none
		0.5%	none	none	none	none

40	6	5%	1	1	1	1
		2%	1	1	1	1
		1%	1	none	none	none
		0.5%	4	none	none	none
50	8	5%	1	1	1	1
		2%	1	1	1	1
		1%	1	none	none	none
		0.5%	3	none	none	none
80	12	5-1%	1	1	1	1
		0.5%	1	1	none	none
100	16	5-1%	1	1	1	1
		0.5%	1	1	1	none
125	20	5-1%		1	1	1
		0.5%		1	1	1
150	24	5- 0.5%		1	1	1
175	28	5- 0.5%		1	1	1
200	40	5- 0.5%		1	1	1

40	4	5%	1	1	1	1
		2%	1	1	1	1
		1%	1	1	1	1
		0.5%	4	4	none	4
50	5	5%	1	1	1	1
		2%	1	1	1	1
		1%	1	1	1	1
		0.5%	3	none	none	3
80	8	5-1%	1	1	1	1
		0.5%	1	1	1	1
100	10	5-1%	1	1	1	1
		0.5%	1	1	1	1
125	12	5-1%		1	1	1
		0.5%		1	none	1
150	15	5- 0.5%		1	1	1
175	17	5- 0.5%		1	1	1
200	20	5- 0.5%		1	1	1

Tables for s_{max} = 10

Table 25. Values (in %) for $\Gamma(n_0; N; s_{min}; s_{max})$ in (4.11).

			s_{max} = 10. s_{min} =			
N	n_0	n_m	0.5	0.2	0.1	0.05
25	1	2	5.9	16.5	15.5	14.0
	2		1.5	16.5	15.5	14.0
40	1	3	3.0	5.1	5.1	5.0
	2		1.8	5.1	5.1	5.0
	3		0.97	5.1	5.1	5.0
50	1	4	2.1	4.9	4.8	4.9
	2		1.4	4.9	4.8	4.9
	3		0.90	4.9	4.8	4.9
	4		0.82	4.9	4.8	4.9
80	1	7	0.99	1.6	4.2	4.5
	2		0.78	1.6	4.2	4.5
	3		0.58	1.6	4.2	4.5
	4		0.41	1.6	4.2	4.5
	5		0.36	1.6	4.2	4.5
100	1	9	0.67	1.1	4.2	4.2
	2		0.56	1.1	4.2	4.2
	3		0.45	1.1	4.2	4.2
	4		0.35	1.1	4.2	4.2
	5		0.26	1.1	4.2	4.2
125	1-5	12		0.70	1.9	10.9
150	1-5	14		0.48	1.8	3.8
175	1-5	17		0.36	1.5	5.3
200	1	19		0.28	0.95	3.9

Table 26. Values (in %) for $\Gamma_{0.9}(n_0; N; s_{min}; s_{max})$ in (4.13).

			s_{max} = 10. s_{min} =			
N	n_0	$n_{m,0.9}$	0.5	0.2	0.1	0.05
25	1	2	5.9	16.5	15.5	14.0
	2		1.5	16.5	15.5	14.0
40	1	3	3.0	5.1	5.1	5.0
	2		1.8	5.1	5.1	5.0
	3		0.97	5.1	5.1	5.0
50	1	4	2.1	4.9	4.8	4.9
	2		1.4	4.9	4.8	4.9
	3		0.90	4.9	4.8	4.9
	4		0.82	4.9	4.8	4.9
80	1	7	0.99	1.6	4.2	4.5
	2		0.78	1.6	4.2	4.5
	3		0.58	1.6	4.2	4.5
	4		0.41	1.6	4.2	4.5
	5		0.36	1.6	4.2	4.5
100	1	9	0.67	1.1	4.2	4.2
	2		0.56	1.1	4.2	4.2
	3		0.45	1.1	4.2	4.2
	4		0.35	1.1	4.2	4.2
	5		0.26	1.1	4.2	4.2
125	1- 5	11		0.70	1.9	3.1
150	1- 5	13		0.48	1.3	1.7
175	1- 5	15		0.36	1.0	1.1
200	1	18		0.28	0.95	1.9

Table 27. Values (in %) for $\Gamma_{0.8}(n_0; N; s_{min}; s_{max})$ in (4.13).

			s_{max} = 10. s_{min} =			
N	n_0	$n_{m,0.8}$	0.5	0.2	0.1	0.05
25	1	2	5.9	16.5	15.5	14.0
	2		1.5	16.5	15.5	14.0

Table 28. Values (in %) for $\Gamma_{0.5}(n_0; N; s_{min}; s_{max})$ in (4.13).

			s_{max} = 10. s_{min} =			
N	n_0	$n_{m,0.5}$	0.5	0.2	0.1	0.05
25	1	1	5.9	5.9	6.1	14.0
	2		-	-	-	-

	1		3.0	5.1	5.1	5.0
40	2	3	1.8	5.1	5.1	5.0
	3		0.97	5.1	5.1	5.0
	1		2.1	4.9	4.8	4.9
50	2	4	1.4	4.9	4.8	4.9
	3		0.90	4.9	4.8	4.9
	4		0.82	4.9	4.8	4.9
	1		0.99	1.3	2.1	1.6
	2		0.78	1.3	2.1	1.6
80	3	6	0.58	1.3	2.1	1.6
	4		0.41	1.3	2.1	1.6
	5		0.36	1.3	2.1	1.6
	1		0.67	1.1	2.2	1.6
	2		0.56	1.1	2.2	1.6
100	3	8	0.45	1.1	2.2	1.6
	4		0.35	1.1	2.2	1.6
	5		0.26	1.1	2.2	1.6
	1			0.70	1.1	1.3
	2			0.70	1.1	1.3
125	3	10		0.70	1.1	1.3
	4			0.70	1.1	1.3
	5			0.70	1.1	1.3
	1			0.48	0.81	1.0
	2			0.48	0.81	1.0
150	3	12		0.48	0.81	1.0
	4			0.48	0.81	1.0
	5			0.48	0.81	1.0
	1			0.35	0.68	0.65
	2			0.35	0.68	0.65
175	3	14		0.35	0.68	0.65
	4			0.35	0.68	0.65
	5			0.35	0.68	0.65
	1			0.27	0.54	0.71
	2			0.27	0.54	0.71
200	3	16		0.27	0.54	0.71
	4			0.27	0.54	0.71
	5			0.27	0.54	0.71

	1		3.0	3.1	3.0	3.0
40	2	2	1.8	1.7	2.0	1.7
	1		2.1	2.1	2.1	2.1
50	2	2	1.4	2.1	1.4	1.4
	1		0.99	0.99	0.99	0.98
	2		0.78	0.78	0.98	078
80	3	4	0.58	0.71	0.98	0.58
	4		0.41	0.71	0.98	0.56
	1		0.67	0.67	0.67	0.67
	2		0.56	0.59	0.58	0.56
100	3	5	0.45	0.59	0.58	0.47
	4		0.35	0.59	0.58	0.47
	5		0.26	0.59	0.58	0.47
	1			0.45	0.45	0.45
	2			0.39	0.39	0.39
125	3	6		0.37	0.37	0.33
	4			0.37	0.37	0.28
	5			0.37	0.37	0.22
	1			0.33	0.33	0.33
	2			0.29	0.29	0.29
150	3	7		0.25	0.25	0.26
	4			0.25	0.22	0.26
	5			0.25	0.19	0.26
	1			0.25	0.25	0.25
	2			0.22	0.22	0.22
175	3	8		0.20	0.20	0.20
	4			0.18	0.18	0.18
	5			0.17	0.15	0.15
	1			0.19	0.19	0.19
	2			0.18	0.18	0.18
200	3	10		0.16	0.17	0.16
	4			0.14	0.17	0.14
	5			0.14	0.17	0.13

Table 29. β - sufficient sample sizes.

			s_{max} = 10. s_{min} =			
N	n_m	β	0.5	0.2	0.1	0.05
25	2	5-2%	2	none	none	none
		1-0.5%	none	none	none	none
		5%	1	none	none	none
40	3	2%	2	none	none	none
		1%	3	none	none	none
		0.5%	none	none	none	none
		5%	1	1	1	1
50	4	2%	2	none	none	none
		1%	3	none	none	none
		0.5%	none	none	none	none
		5%	1	1	1	1
80	7	2%	1	1	none	none
		1%	1	none	none	none
		0.5%	4	none	none	none
		5%	1	1	1	1
100	9	2-1%	1	1	none	none
		0.5%	3	9	none	none

Table 30. (β, 0.9) - sufficient sample sizes.

			s_{max} = 10. s_{min} =			
N	$n_{m,0.9}$	β	0.5	0.2	0.1	0.05
25	2	5-2%	2	none	none	none
		1-0.5%	none	none	none	none
		5%	1	none	none	none
40	3	2%	2	none	none	none
		1%	3	none	none	none
		0.5%	none	none	none	none
		5%	1	1	1	1
50	4	2%	2	none	none	none
		1%	3	none	none	none
		0.5%	none	none	none	none
		5%	1	1	1	1
80	7	2%	1	1	none	none
		1%	1	none	none	none
		0.5%	4	none	none	none
		5%	1	1	1	1
100	9	2-1%	1	1	none	none
		0.5%	3	9	none	none

N	n_m	β	0.5	0.2	0.1	0.05
125	12	5%		1	1	none
		2%		1	1	none
		1%		1	none	none
		0.5%		12	none	none
150	14	5%		1	1	1
		2%		1	1	none
		1 - 0.5%		1	none	none
175	17	5 - 2%		1	1	none
		1 - 0.5%		1	none	none
200	19	5%		1	1	1
		2%		1	1	none
		1%		1	1	none
		0.5%		1	none	none

N	$n_{m,0.5}$	β	0.5	0.2	0.1	0.05
125	11	5%		1	1	1
		2%		1	1	none
		1%		1	none	none
		0.5%		none	none	none
150	13	5%		1	1	1
		2%		1	1	1
		1 - 0.5%		1	none	none
175	15	5 - 2%		1	1	1
		1 - 0.5%		1	none	none
200	18	5%		1	1	1
		2%		1	1	1
		1%		1	1	none
		0.5%		1	none	none

Table 31. (β, 0.8) - sufficient sample sizes.

			$s_{max} = 10$. s_{min} =			
N	n_m	β	0.5	0.2	0.1	0.05
25	2	5 - 2%	2	none	none	none
		1 - 0.5%	none	none	none	none
40	3	5%	1	none	none	none
		2%	2	none	none	none
		1%	3	none	none	none
		0.5%	none	none	none	none
50	4	5%	1	1	1	1
		2%	2	none	none	none
		1%	3	none	none	none
		0.5%	none	none	none	none
80	6	5%	1	1	1	1
		2%	1	1	none	1
		1%	1	none	none	none
		0.5%	4	none	none	none
100	8	5%	1	1	1	1
		2%	1	1	none	1
		1%	1	1	none	none
		0.5%	3	none	none	none
125	10	5 - 2%		1	1	1
		1%		1	none	none
		0.5%		none	none	none
150	12	5 - 2%		1	1	1
		1 - 0.5%		1	none	none
175	14	5 - 1%		1	1	1
		0.5%		1	none	none
200	16	5 - 1%		1	1	1
		0.5%		1	none	none

Table 32. (β, 0.5) - sufficient sample sizes.

			$s_{max} = 10$. s_{min} =			
N	$n_{m,0.5}$	β	0.5	0.2	0.1	0.05
25	1	5%	none	none	none	none
		1 - 0.5%	none	none	none	none
40	2	5%	1	1	1	1
		2%	2	none	none	2
		1%	none	none	none	none
		0.5%	none	none	none	none
50	2	5%	1	1	1	1
		2%	2	2	2	2
		1%	none	none	none	none
		0.5%	none	none	none	none
80	4	5%	1	1	1	1
		2%	1	1	1	1
		1%	1	1	1	1
		0.5%	4	none	none	none
100	5	5%	1	1	1	1
		2%	1	1	1	1
		1%	1	1	1	1
		0.5%	3	none	none	3
125	6	5 - 2%		1	1	1
		1%		1	1	1
		0.5%		1	1	1
150	7	5 - 2%		1	1	1
		1 - 0.5%		1	1	1
175	8	5 - 1%		1	1	1
		0.5%		1	1	1
200	10	5%		1	1	1
		0.5%		1	1	1

Tables for weak quantities, s_{max} = 2 and 4

Table 33. Values (in %) for $w\Gamma(n_0; N; s_{min}; s_{max})$.

			$s_{max} = 2$. s_{min} =			
N	n_0	n_m	0.5	0.2	0.1	0.05
25	1	12	0.28	0.27	0.27	0.28
	2		0.25	0.25	0.26	0.28
	3		0.23	0.23	0.26	0.28
	4		0.20	0.23	0.26	0.28
	5		0.18	0.23	0.26	0.28

Table 34. Values (in %) for $w\Gamma_{0.9}(n_0; N; s_{min}; s_{max})$.

			$s_{max} = 4$. s_{min} =			
N	n_0	$n_{m,0.5}$	0.5	0.2	0.1	0.05
25	1	6	1.3	1.3	1.3	1.3
	2		1.0	1.0	1.0	1.0
	3		0.69	0.69	0.69	0.69
	4		0.42	0.43	0.47	0.44
	5		0.31	0.34	0.43	0.40

40	1	19	0.11	0.11	0.11	0.11
	2		0.11	0.11	0.11	0.11
	3		0.10	0.10	0.10	0.10
	4		0.10	0.10	0.10	0.10
	5		0.09	0.09	0.10	0.09
50	1	24	0.07	0.07	0.07	0.07
	2		0.07	0.07	0.07	0.07
	3		0.07	0.07	0.07	0.07
	4		0.06	0.07	0.06	0.07
	5		0.06	0.06	0.06	0.07
80	1	39	0.03	0.03	0.03	0.03
	2		0.03	0.03	0.03	0.03
	3		0.03	0.03	0.03	0.03
	4		0.03	0.03	0.03	0.03
	5		0.03	0.03	0.03	0.03
100	1	49	0.02	0.02	0.02	0.02
	2		0.02	0.02	0.02	0.02
	3		0.02	0.02	0.02	0.02
	4		0.02	0.02	0.02	0.02
	5		0.02	0.02	0.02	0.02

40	1	9	0.58	0.58	0.58	0.58
	2		0.50	0.50	0.50	0.50
	3		0.41	0.41	0.41	0.41
	4		0.33	0.33	0.33	0.33
	5		0.25	0.25	0.25	0.25
50	1	12	0.39	0.39	0.39	0.39
	2		0.35	0.34	0.34	0.34
	3		0.30	0.30	0.30	0.30
	4		0.26	0.26	0.26	0.26
	5		0.21	0.21	0.21	0.21
80	1	19	0.16	0.16	0.16	0.16
	2		0.15	0.15	0.15	0.15
	3		0.14	0.14	0.14	0.14
	4		0.13	0.13	0.13	0.13
	5		0.12	0.12	0.12	0.12
100	1	24	0.11	0.11	0.11	0.11
	2		0.10	0.10	0.10	0.10
	3		0.10	0.10	0.10	0.10
	4		0.09	0.09	0.09	0.09
	5		0.08	0.08	0.08	0.08

Table 35. Weakly sufficient sample sizes.

N	n_m	β	$s_{max}=2$. $s_{min}=$ 0.5	0.2	0.1	0.05
25	12	5-2%	1	1	1	1
		1%	1	1	1	1
		0.5%	1	1	1	1
40	19	5-1%	1	1	1	1
		0.5%	1	1	1	1
50	24	5-0.5%	1	1	1	1
80	39	5-0.5%	1	1	1	1
100	49	5-0.5%	1	1	1	1

Table 36. Weakly sufficient sample sizes.

N	$n_{m,0.9}$	β	$s_{max}=4$. $s_{min}=$ 0.5	0.2	0.1	0.05
25	6	5-2%	1	1	1	1
		1%	3	3	3	3
		0.5%	4	4	4	4
40	9	5-1%	1	1	1	1
		0.5%	2	2	2	2
50	12	5-0.5%	1	1	1	1
80	19	5-0.5%	1	1	1	1
100	24	5-0.5%	1	1	1	1

Tables for weak quantities, s_{max} = 5 and 10

Table 37. Values (in %) for $w\Gamma(n_0; N; s_{min}; s_{max})$.

N	n_0	$n_{m,0.8}$	$s_{max}=5$. $s_{min}=$ 0.5	0.2	0.1	0.05
25	1	4	2.0	2.0	2.0	2.0
	2		1.4	1.4	1.4	1.4
	3		0.80	0.78	0.77	0.82
	4		0.57	0.57	0.58	0.74
40	1	7	0.91	0.91	0.91	0.91
	2		0.73	0.73	0.73	0.73
	3		0.56	0.56	0.56	0.56
	4		0.40	0.40	0.40	0.40
	5		0.26	0.28	0.28	0.28
50	1	9	0.61	0.61	0.61	0.61
	2		0.52	0.52	0.52	0.52
	3		0.43	0.43	0.43	0.43
	4		0.34	0.34	0.34	0.34
	5		0.26	0.25	0.25	0.25

Table 38. Values (in %) for $w\Gamma(n_0; N; s_{min}; s_{max})$.

N	n_0	$n_{m,0.5}$	$s_{max}=10$. $s_{min}=$ 0.5	0.2	0.1	0.05
25	1	2	5.9	5.9	5.9	5.9
	2		1.3	1.3	1.3	1.8
40	1	3	3.0	3.0	3.0	3.0
	2		1.7	1.7	1.7	1.7
	3		0.58	0.80	0.93	1.3
50	1	4	2.1	2.1	2.1	2.1
	2		1.4	1.4	1.4	1.4
	3		0.78	0.78	0.77	0.80
	4		0.57	0.58	0.58	0.64
	5		-	-	-	-

80	1	15	0.26	0.26	0.26	0.26
	2		0.24	0.24	0.24	0.24
	3		0.21	0.21	0.21	0.21
	4		0.19	0.19	0.19	0.19
	5		0.16	0.16	0.16	0.16
100	1	19	0.17	0.17	0.17	0.17
	2		0.16	0.16	0.16	0.16
	3		0.15	0.15	0.15	0.15
	4		0.13	0.13	0.13	0.13
	5		0.12	0.12	0.12	0.12
125	1	24		0.11	0.11	0.11
	2			0.11	0.11	0.11
	3			0.10	0.10	0.10
	4			0.09	0.09	0.09
	5			0.09	0.09	0.09
150	1	29		0.08	0.08	0.08
	2			0.08	0.08	0.08
	3			0.07	0.07	0.07
	4			0.07	0.07	0.07
	5			0.06	0.06	0.06
175	1	34		0.06	0.06	0.06
	2			0.06	0.06	0.06
	3			0.06	0.06	0.06
	4			0.05	0.05	0.05
	5			0.05	0.05	0.05
200	1	39		0.05	0.05	0.05
	2			0.05	0.05	0.05
	3			0.04	0.04	0.04
	4			0.04	0.04	0.04
	5			0.04	0.04	0.04

80	1	7	0.99	0.99	0.99	0.98
	2		0.78	0.78	0.78	0.78
	3		0.58	0.58	0.58	0.58
	4		0.41	0.41	0.41	0.41
	5		0.26	0.28	0.28	0.28
100	1	9	0.67	0.67	0.67	0.67
	2		0.56	0.56	0.56	0.56
	3		0.45	0.45	0.45	0.45
	4		0.35	0.35	0.35	0.35
	5		0.26	0.26	0.26	0.26
125	1	12		0.45	0.45	0.45
	2			0.39	0.39	0.39
	3			0.33	0.33	0.33
	4			0.28	0.28	0.28
	5			0.22	0.22	0.22
150	1	14		0.33	0.33	0.33
	2			0.29	0.29	0.29
	3			0.25	0.25	0.25
	4			0.22	0.22	0.22
	5			0.19	0.19	0.19
175	1	17		0.25	0.25	0.25
	2			0.22	0.22	0.22
	3			0.20	0.20	0.20
	4			0.18	0.18	0.18
	5			0.15	0.15	0.15
200	1	19		0.19	0.19	0.19
	2			0.18	0.18	0.18
	3			0.16	0.16	0.16
	4			0.14	0.14	0.14
	5			0.13	0.13	0.13

Table 39. Weakly sufficient sample sizes.

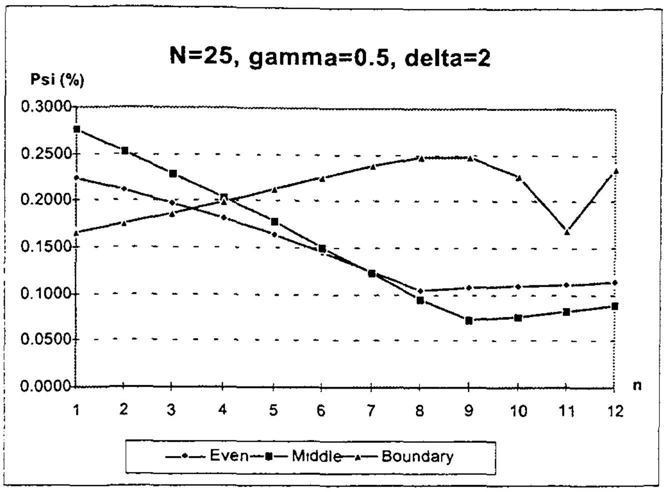
N	n_m	β	$s_{max}=5$. $s_{min}=$ 0.5	0.2	0.1	0.05
25	4	5%	1	1	1	1
		2%	1	1	1	1
		1%	3	3	3	3
		0.5%	none	none	none	none
40	7	5%	1	1	1	1
		2%	1	1	1	1
		1%	1	1	1	1
		0.5%	4	4	4	4
50	9	5%	1	1	1	1
		2%	1	1	1	1
		1%	1	1	1	1
		0.5%	3	3	3	3
80	15	5-1%	1	1	1	1
		0.5%	1	1	1	1
100	19	5-1%	1	1	1	1
		0.5%	1	1	1	1
125	24	5- 0.5%		1	1	1
150	29	5- 0.5%		1	1	1
175	34	5- 0.5%		1	1	1
200	39	5- 0.5%		1	1	1

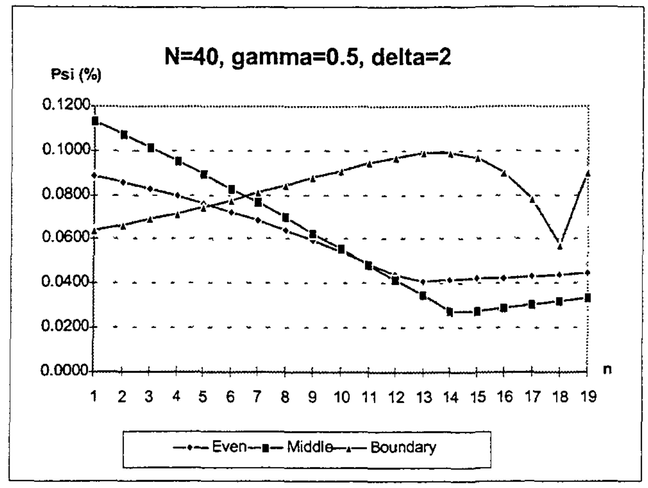
Table 40. Weakly sufficient sample sizes.

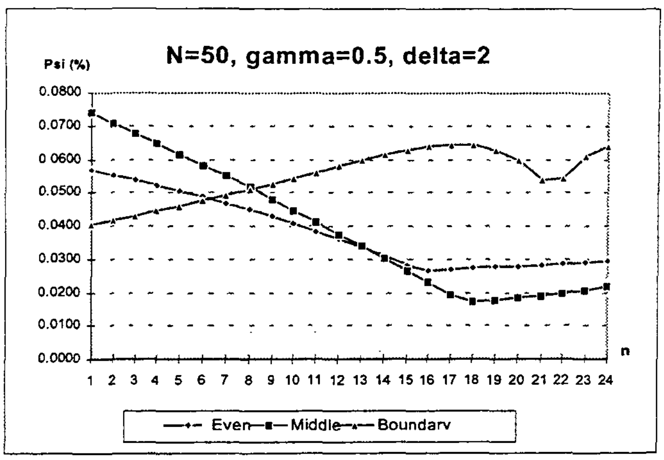
N	$n_{m,0.5}$	β	$s_{max}=10$. $s_{min}=$ 0.5	0.2	0.1	0.05
25	2	5%	2	2	2	2
		2%	2	2	2	2
		1%	none	none	none	none
		0.5%	none	none	none	none
40	3	5%	1	1	1	1
		2%	2	2	2	2
		1%	3	3	3	none
		0.5%	none	none	none	none
50	4	5%	1	1	1	1
		2%	2	2	2	2
		1%	3	3	3	3
		0.5%	none	none	none	none
80	7	5-1%	1	1	1	1
		0.5%	4	4	4	4
100	9	5-1%	1	1	1	1
		0.5%	3	3	3	3
125	12	5- 0.5%		1	1	1
150	14	5- 0.5%		1	1	1
175	17	5- 0.5%		1	1	1
200	19	5- 0.5%		1	1	1

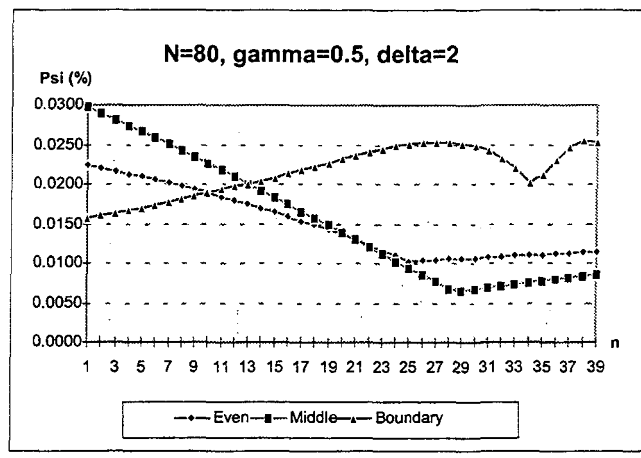
Appendix 2. Ψ-sequence graphs

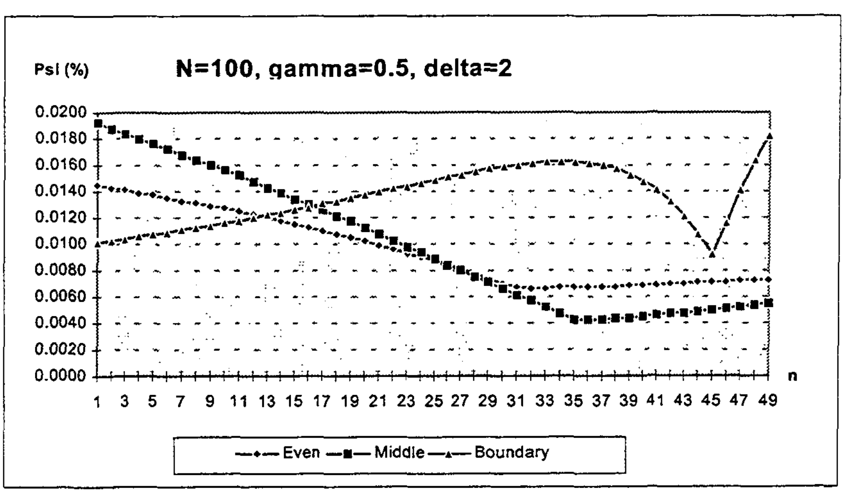
For $s_{max} = \delta = 2$ and $s_{min} = \gamma = 0.5$



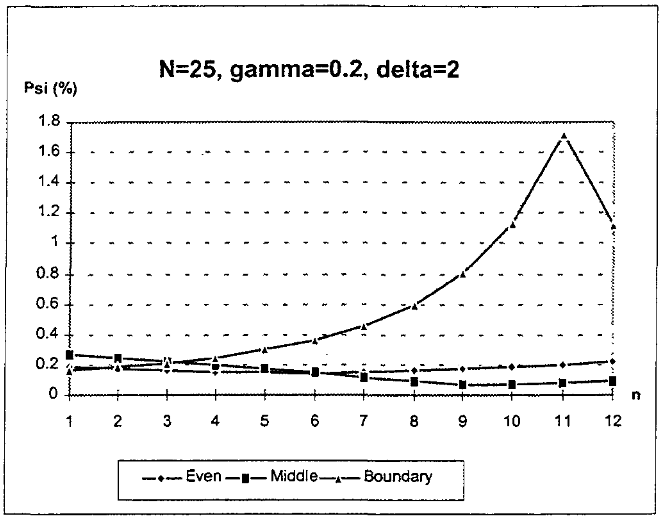


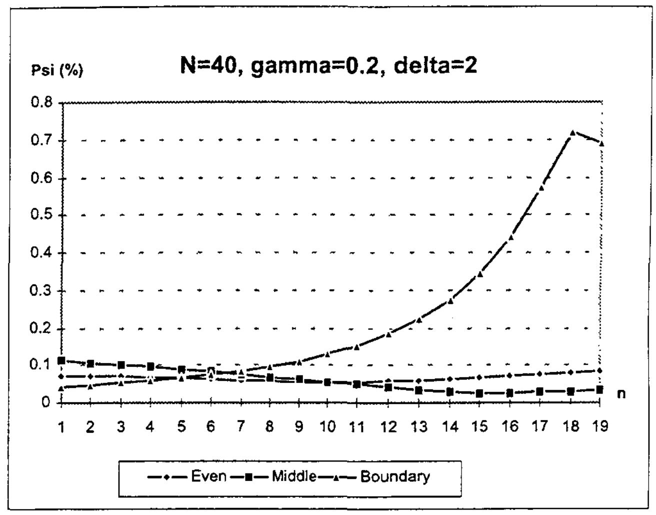


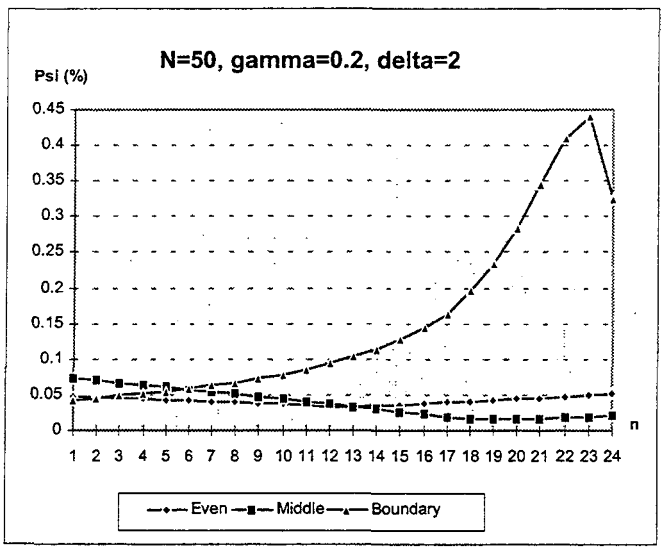


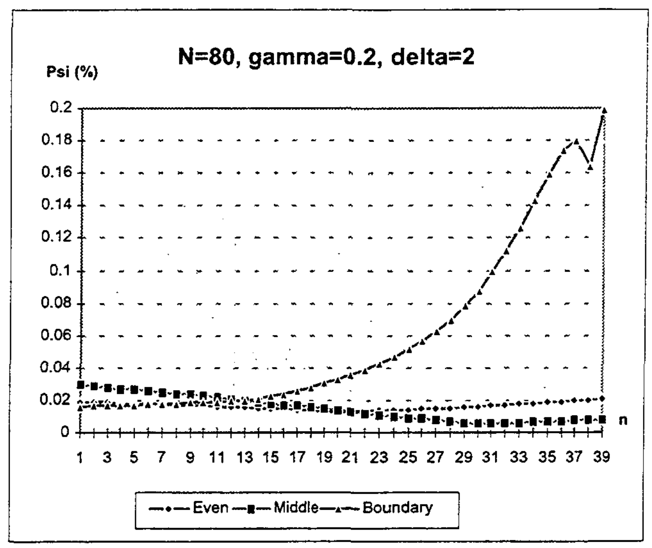


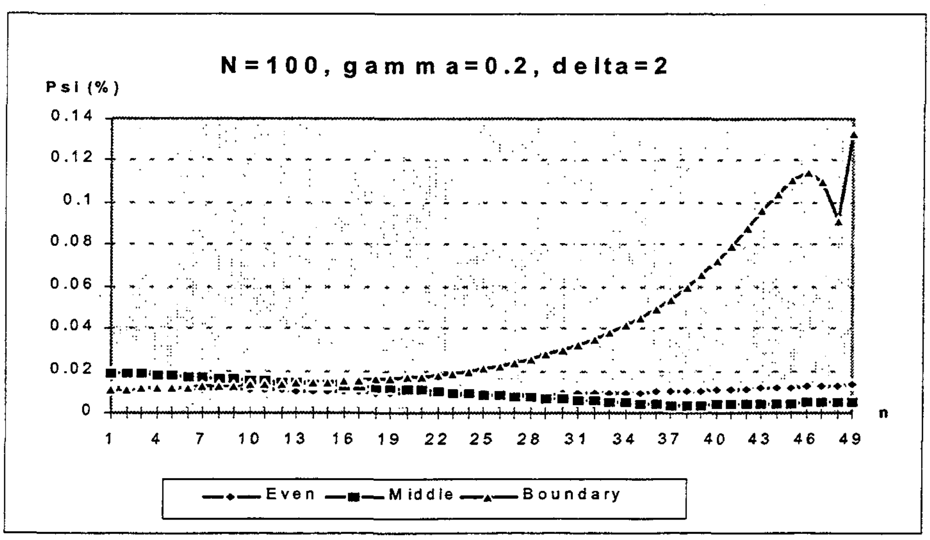
For $s_{max} = \delta = 2$ and $s_{min} = \gamma = 0.2$



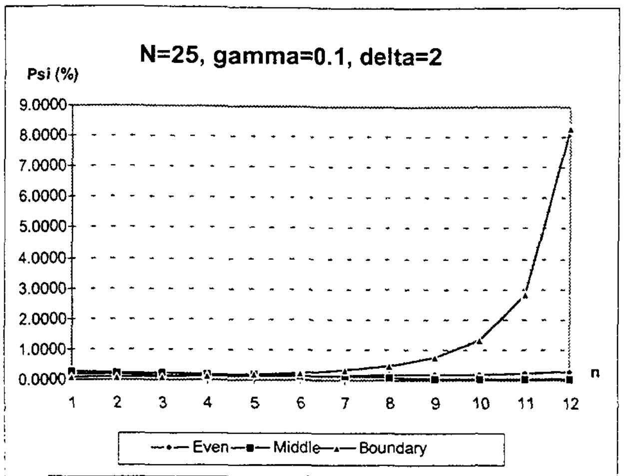


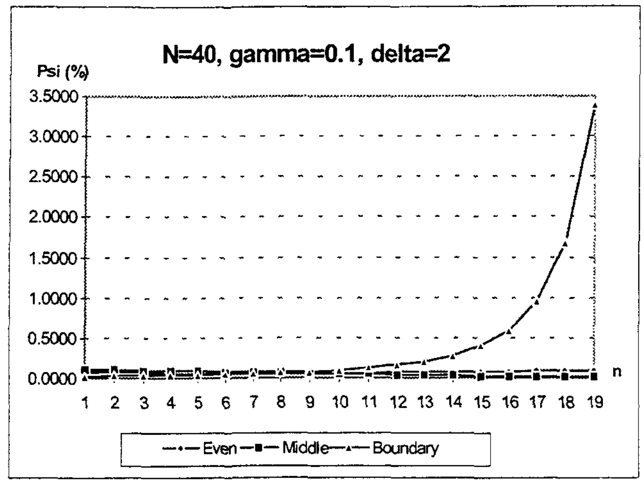


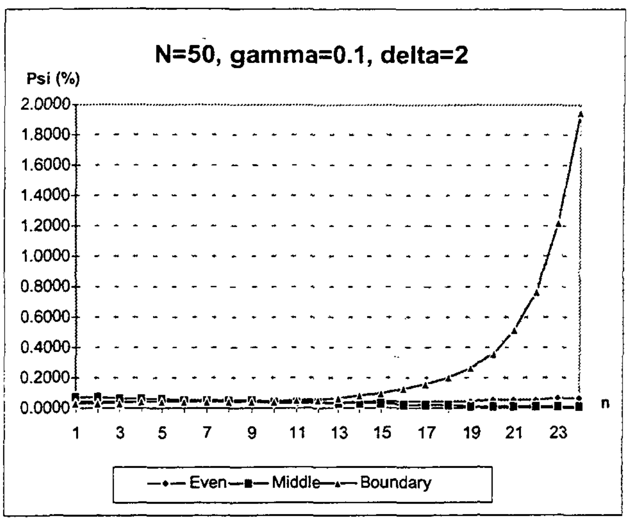


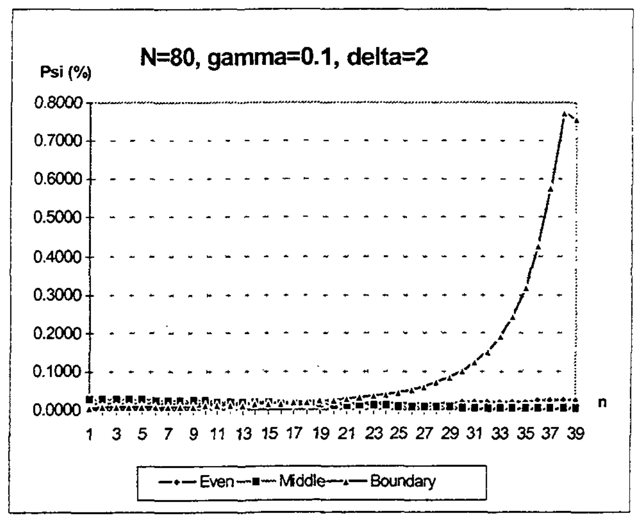


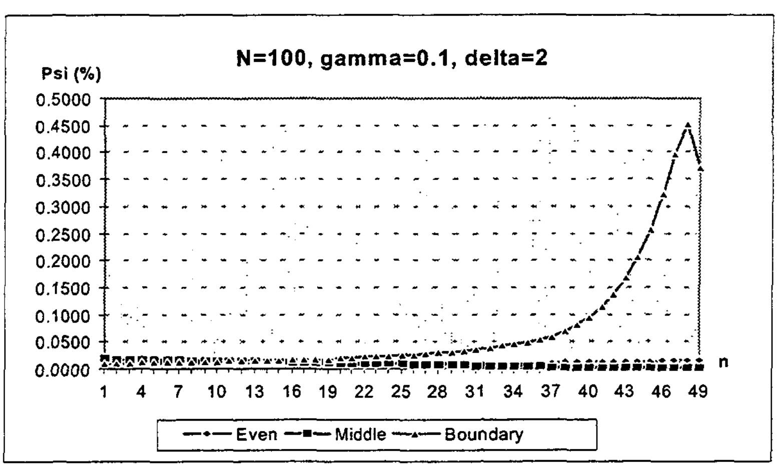
For $s_{max} = \delta = 2$ and $s_{min} = \gamma = 0.1$



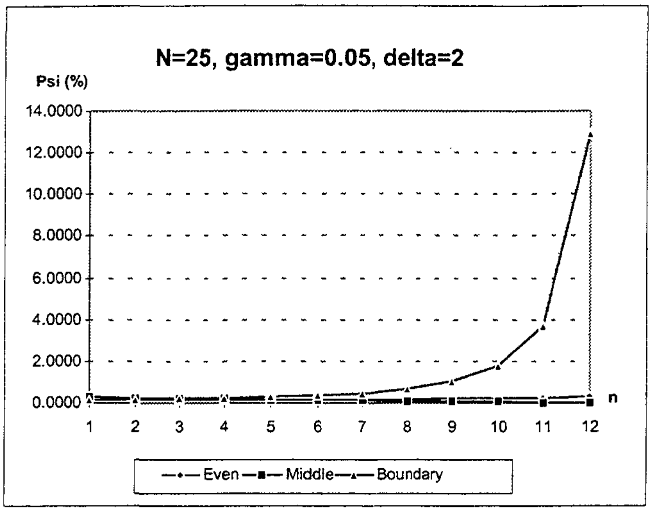


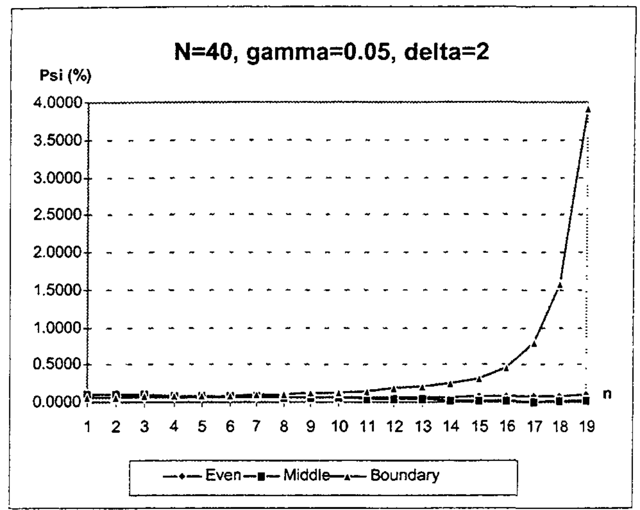


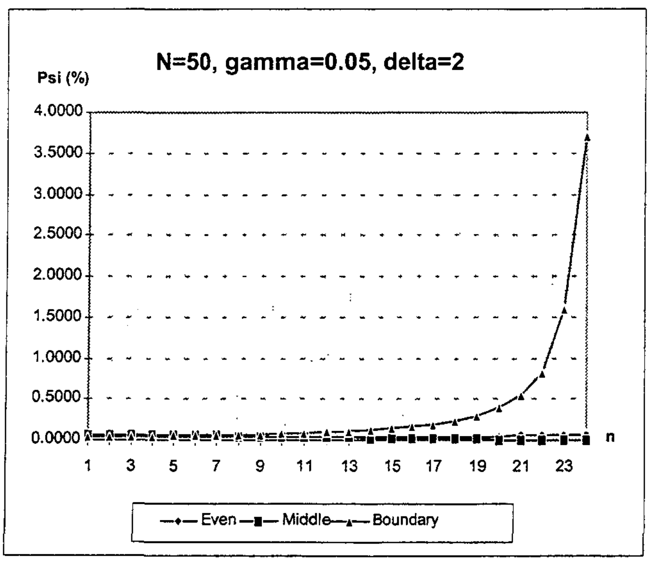


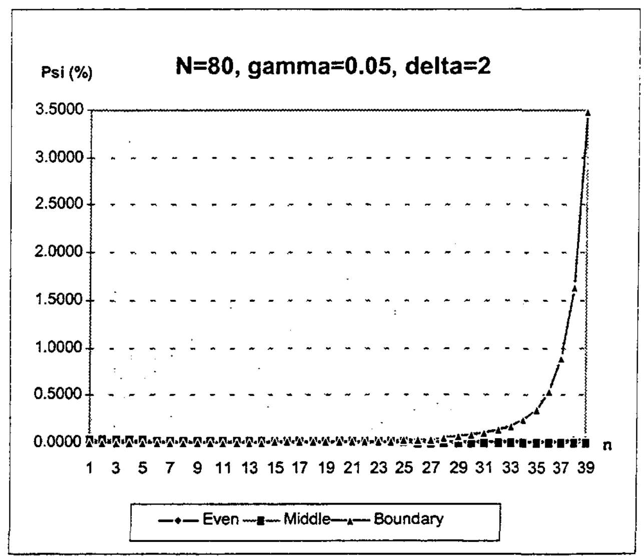


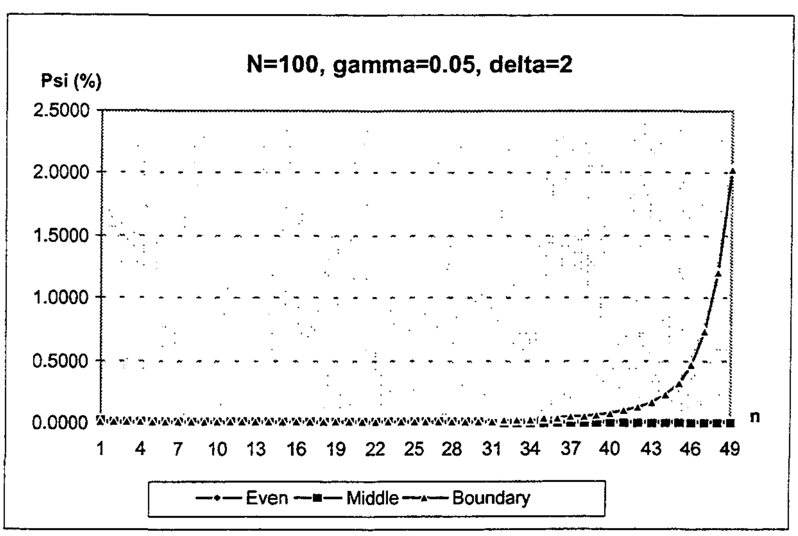
For $s_{max} = \delta = 2$ and $s_{min} = \gamma = 0.05$



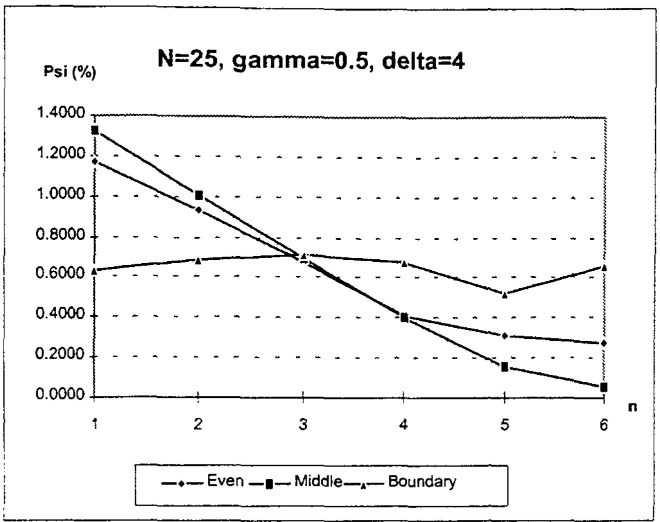


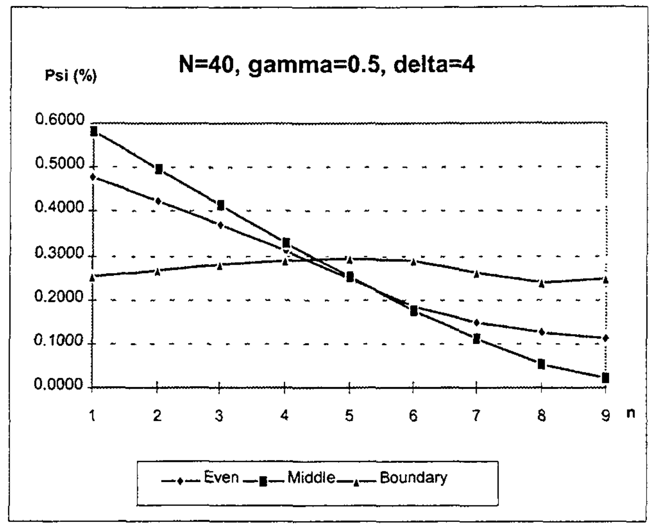


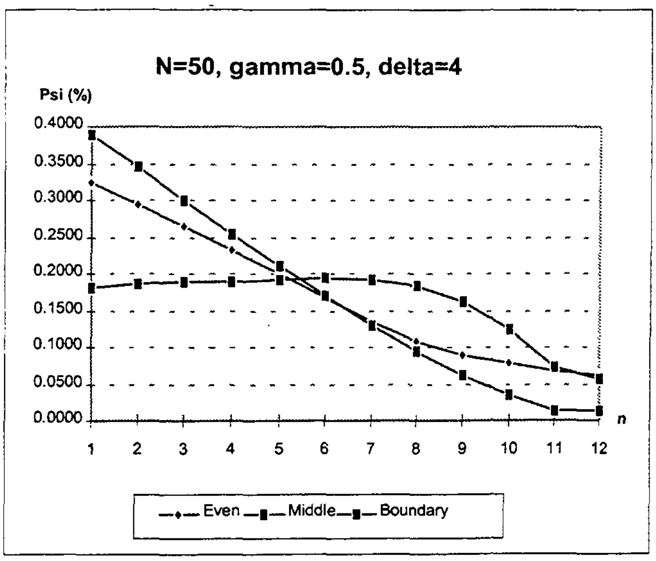


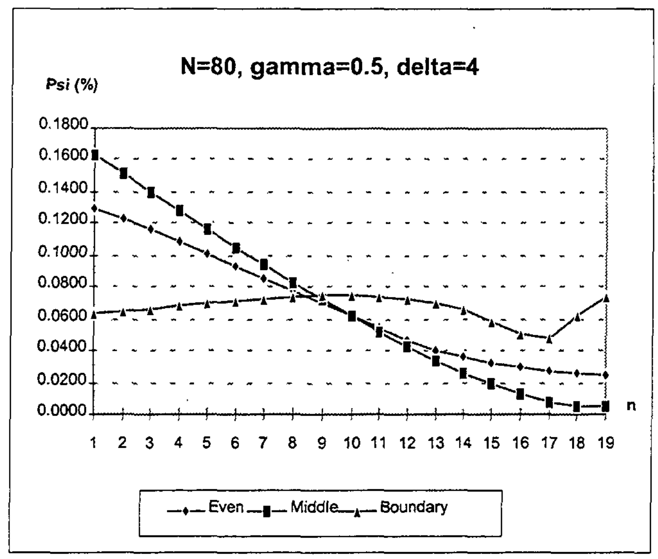


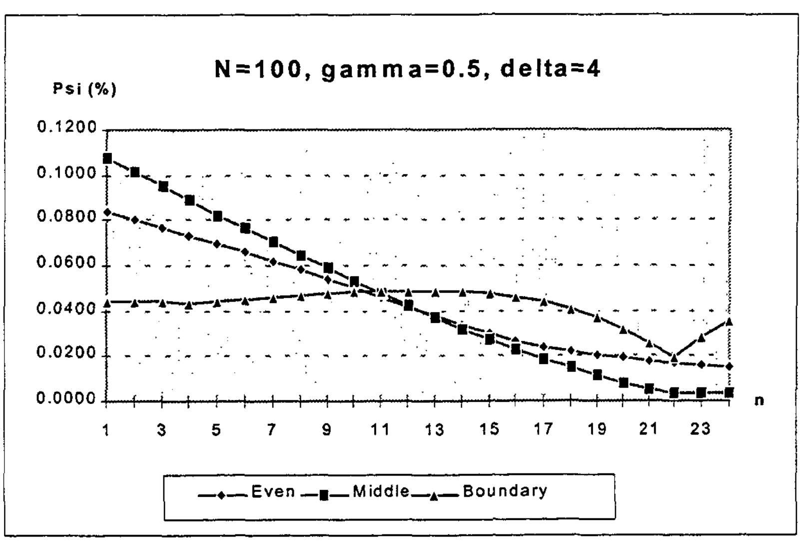
For $s_{max} = \delta = 4$ and $s_{min} = \gamma = 0.5$



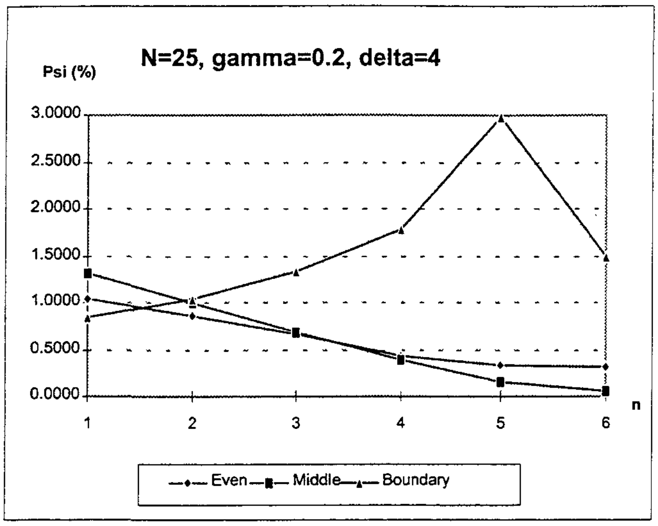


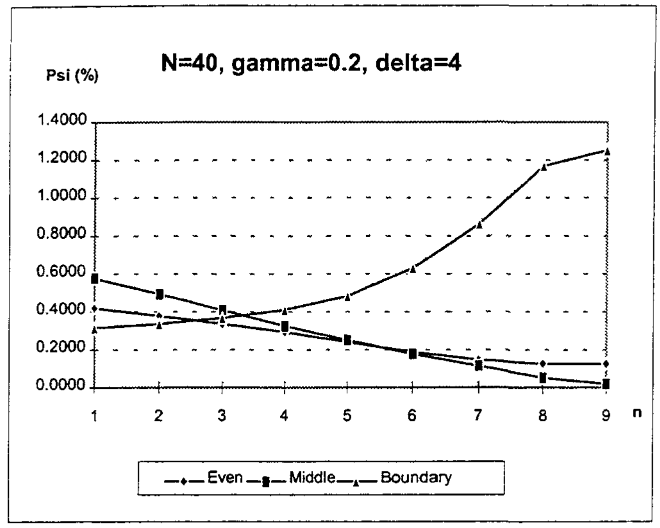


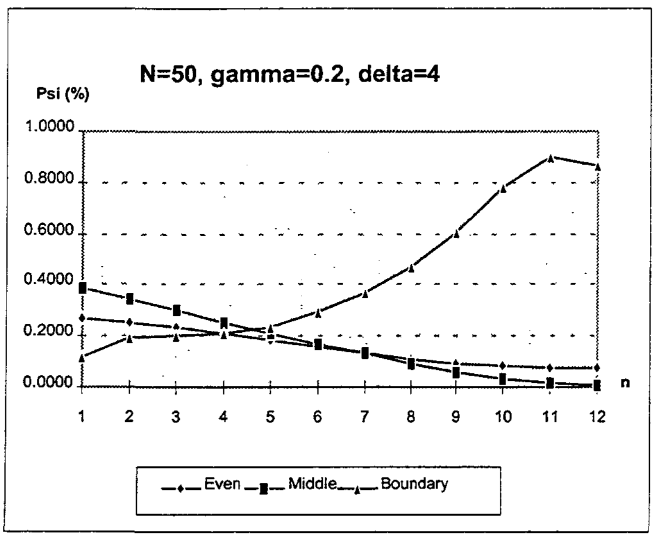


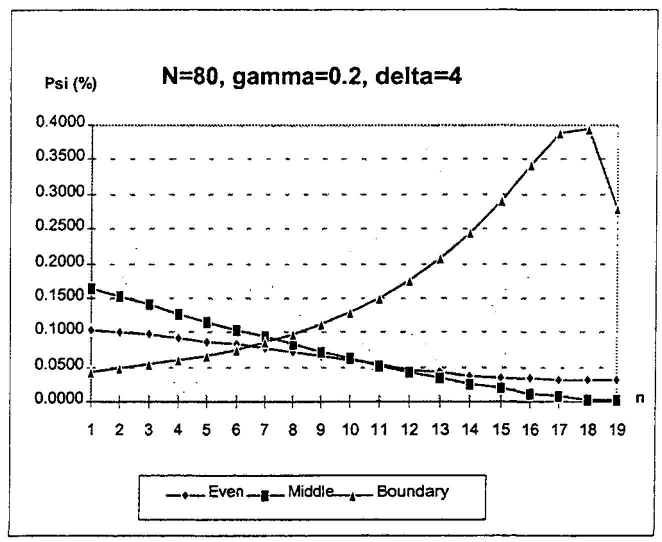


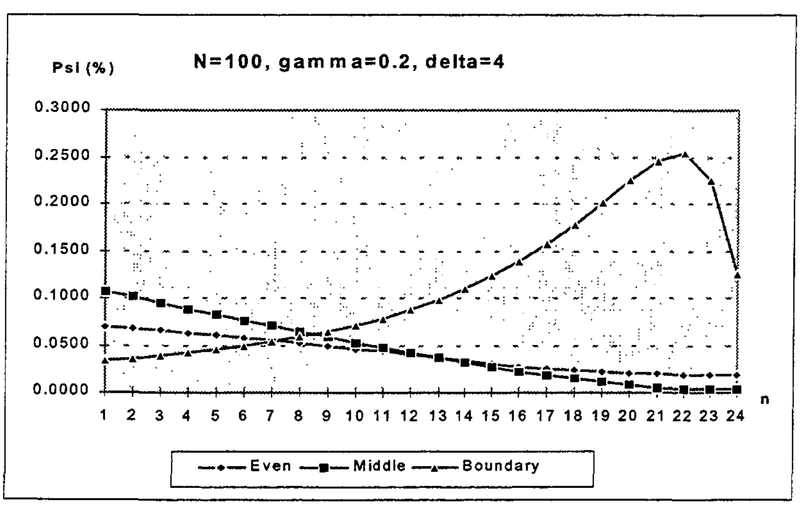
For $s_{max} = \delta = 4$ and $s_{min} = \gamma = 0.2$



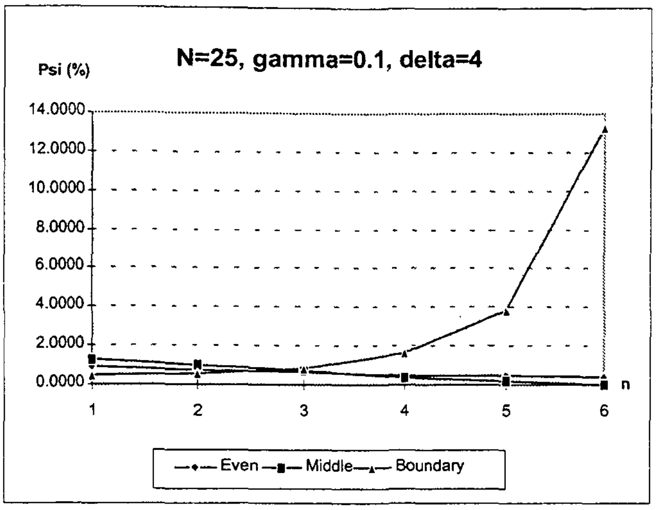


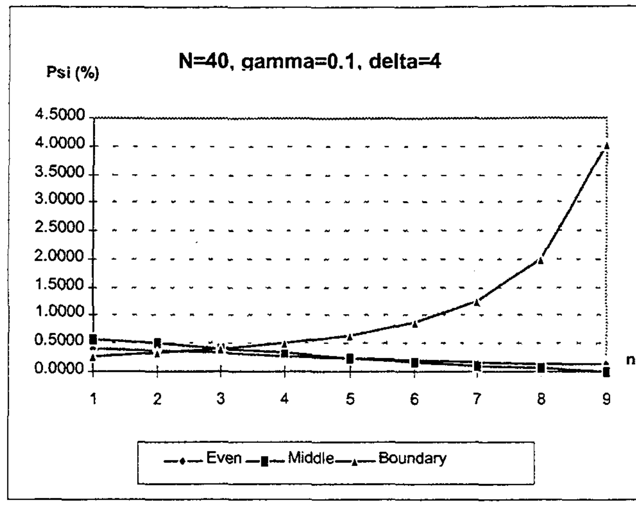


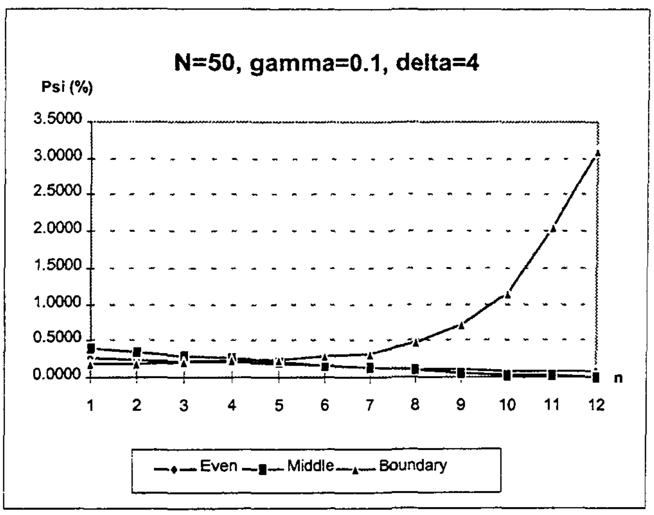


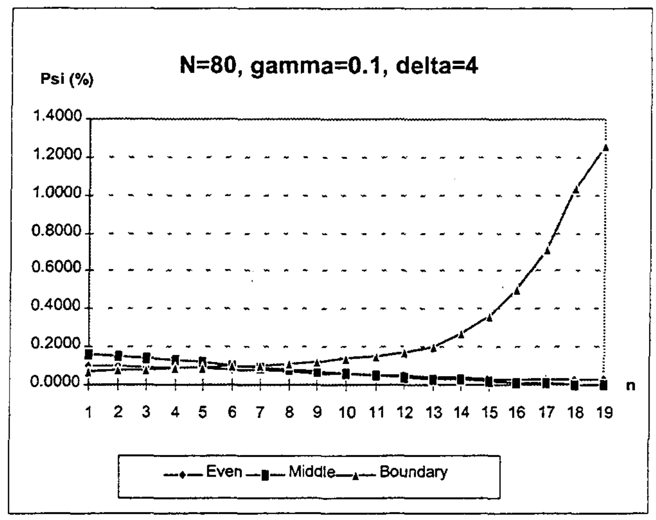


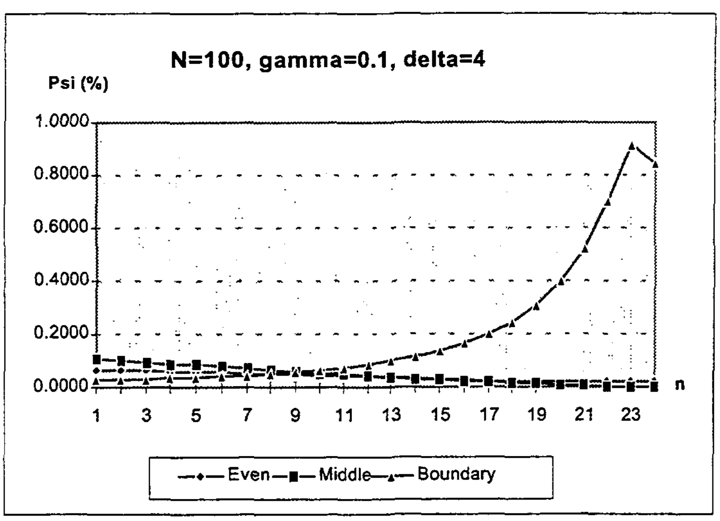
For $s_{max} = \delta = 4$ and $s_{min} = \gamma = 0.1$



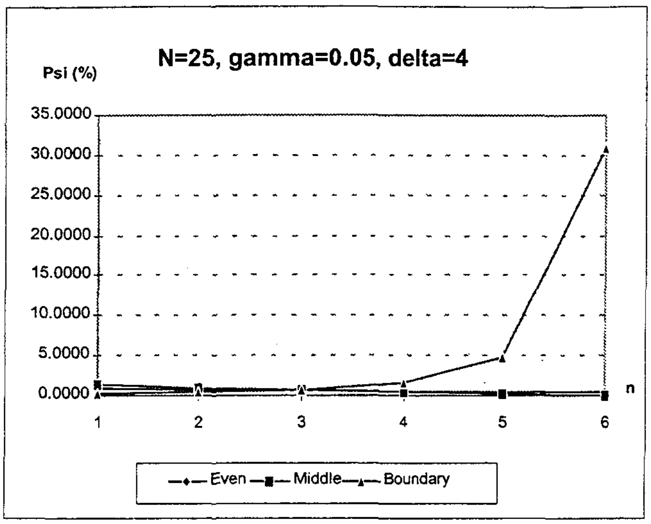


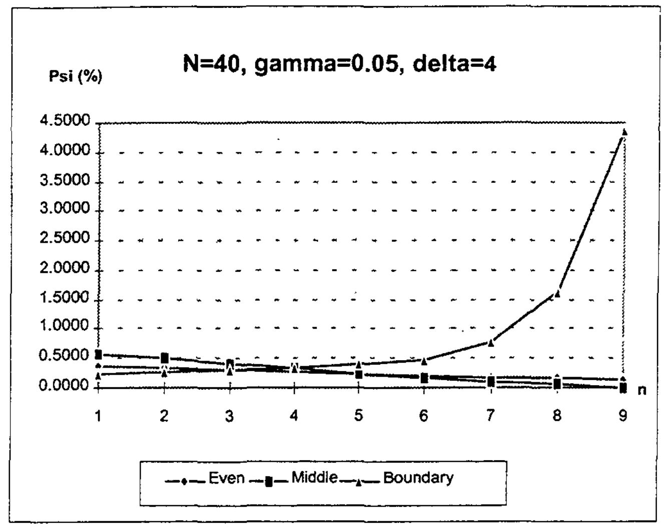


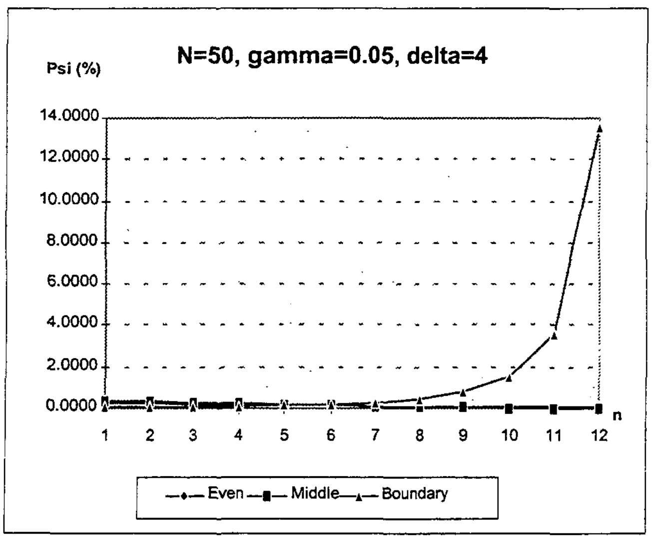


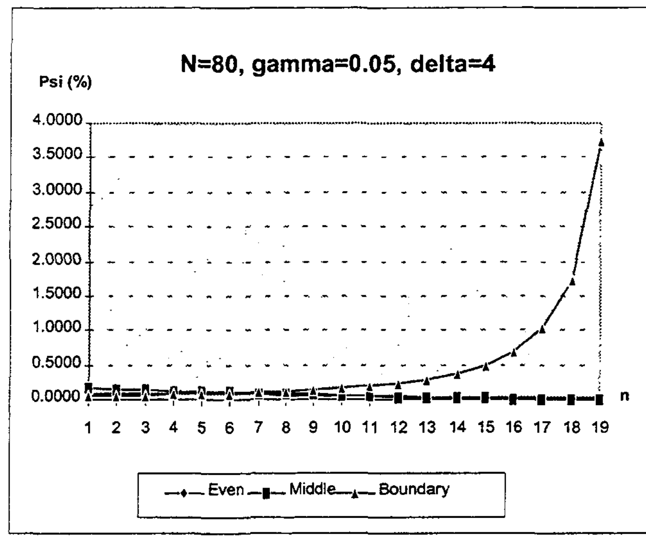


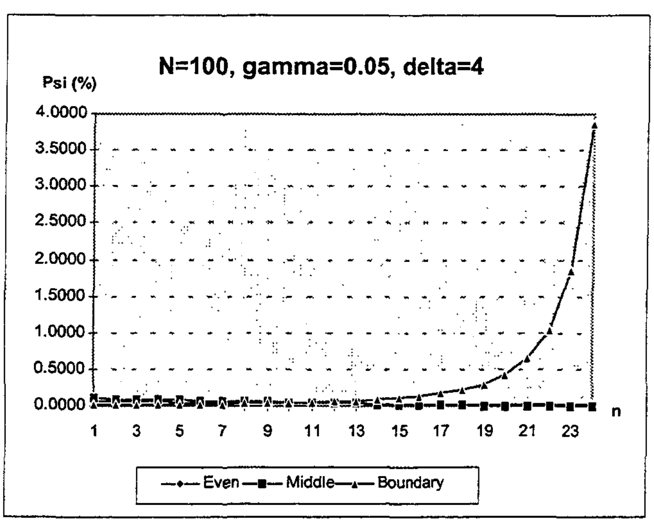
For $s_{max} = \delta = 4$ and $s_{min} = \gamma = 0.05$



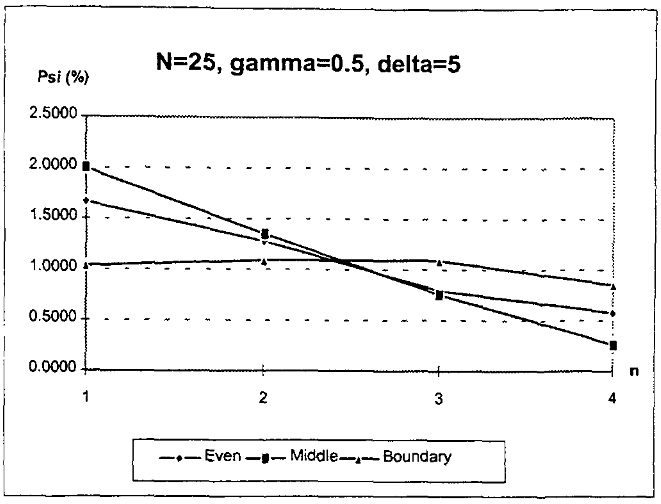


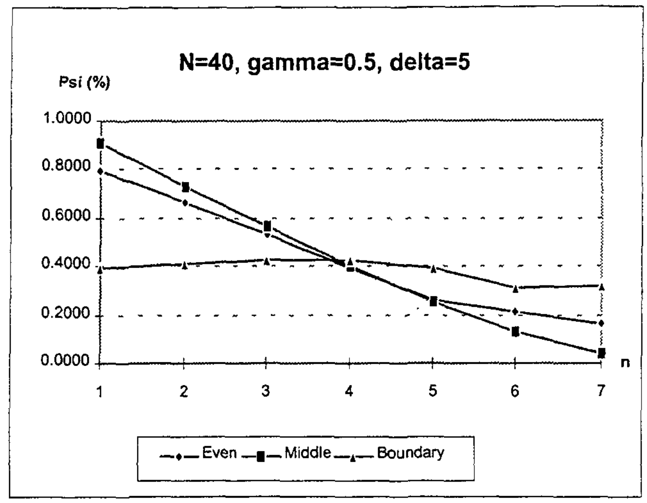


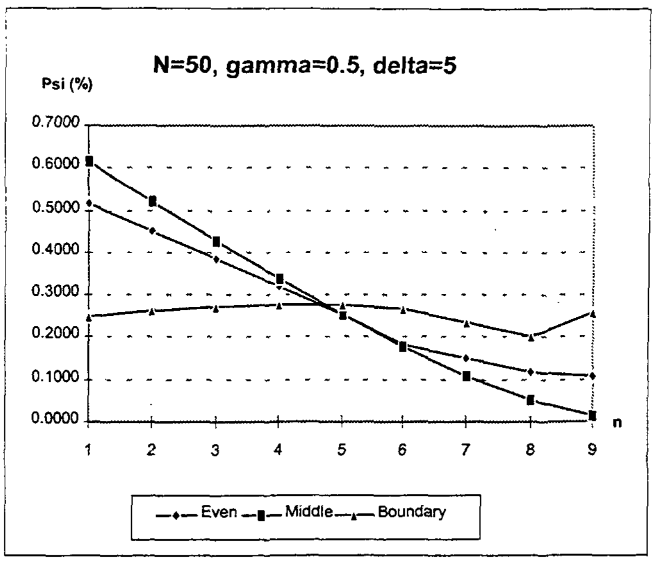


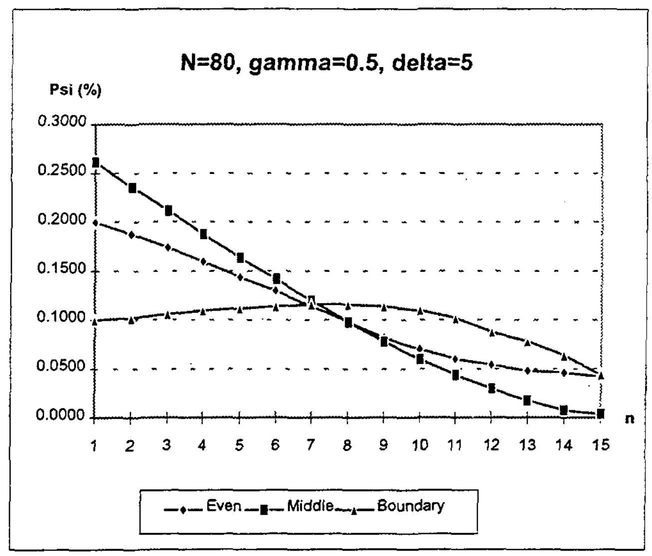


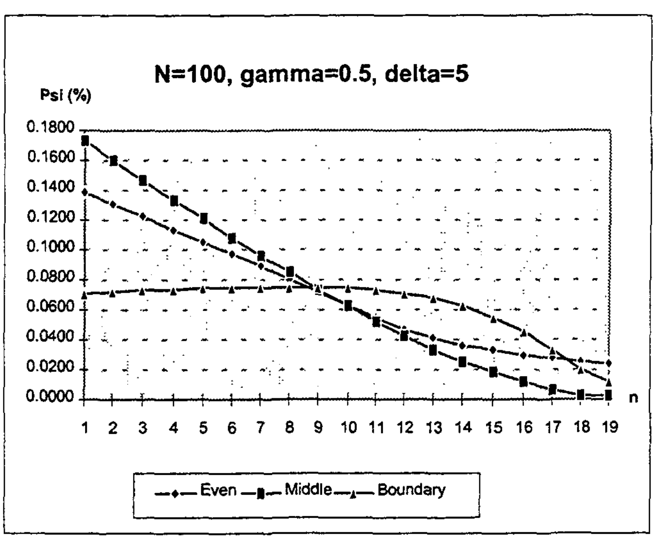
For $s_{max} = \delta = 5$ and $s_{min} = \gamma = 0.5$



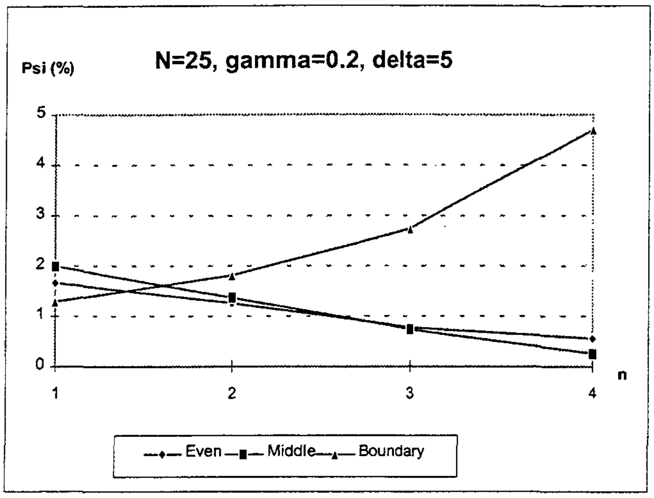


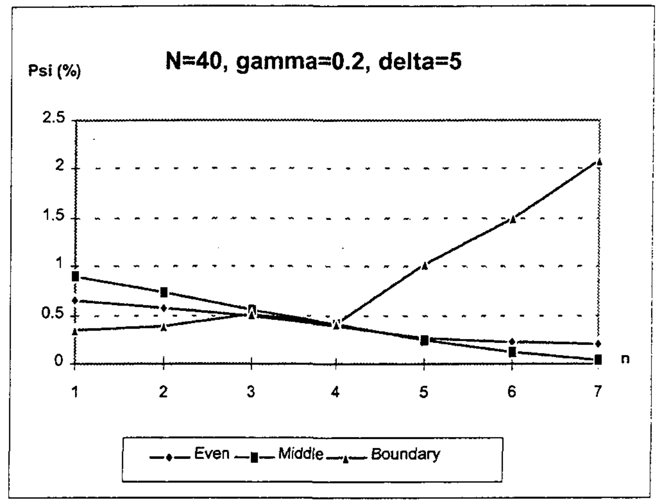


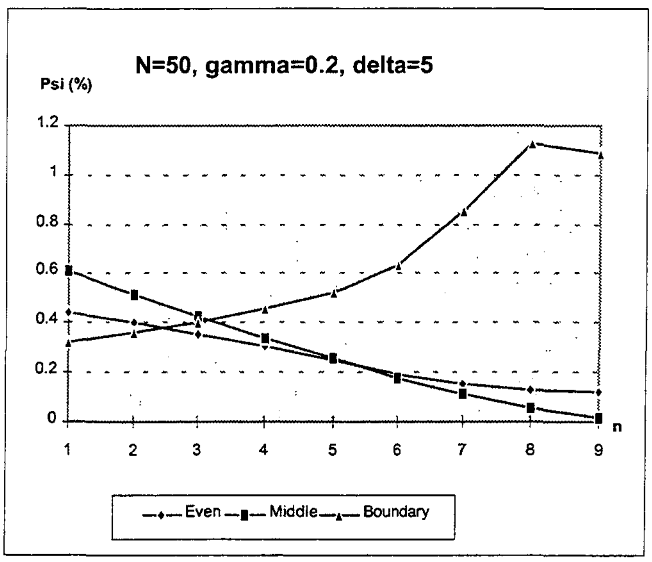


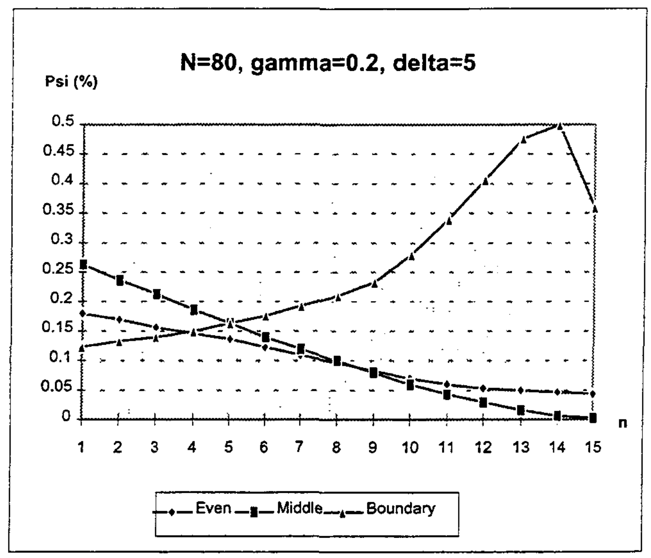


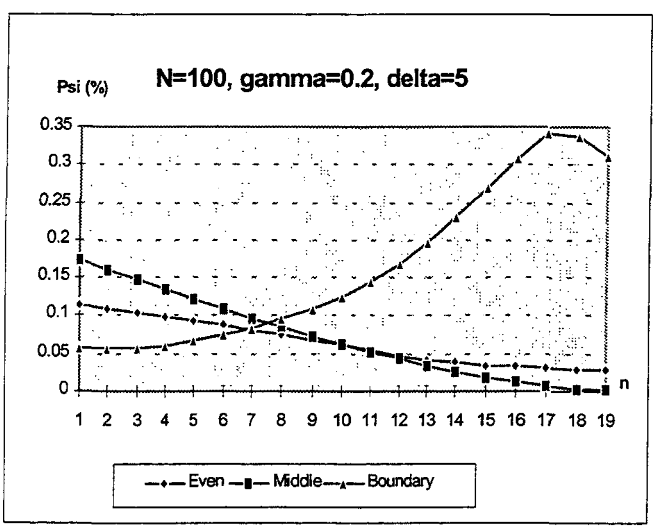
For $s_{max} = \delta = 5$ and $s_{min} = \delta = 0.2$

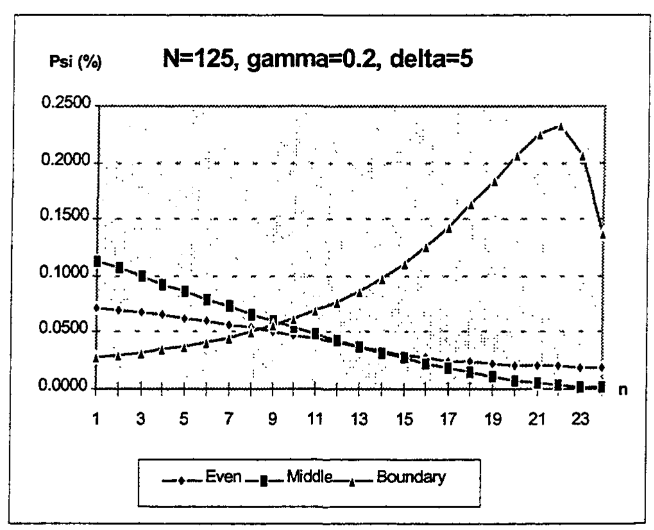


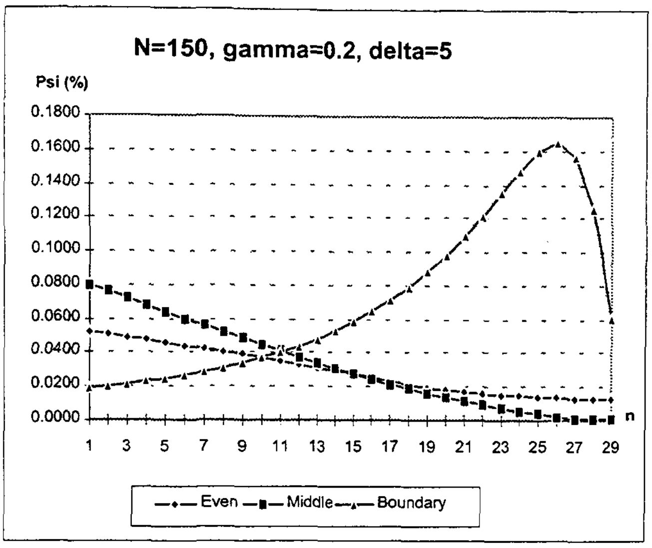


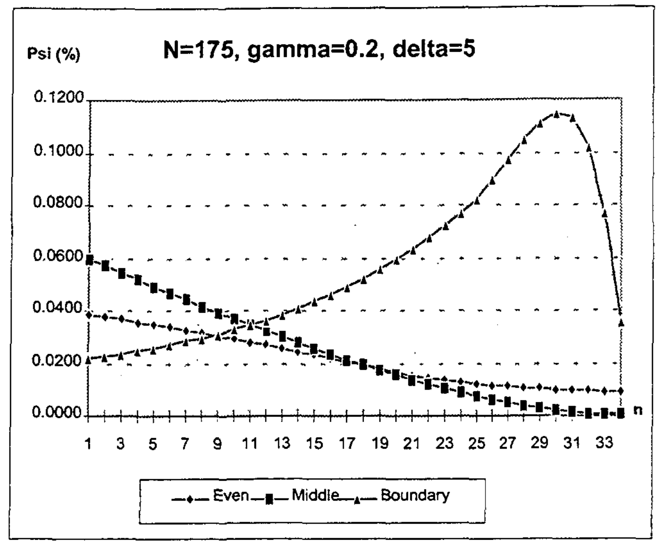


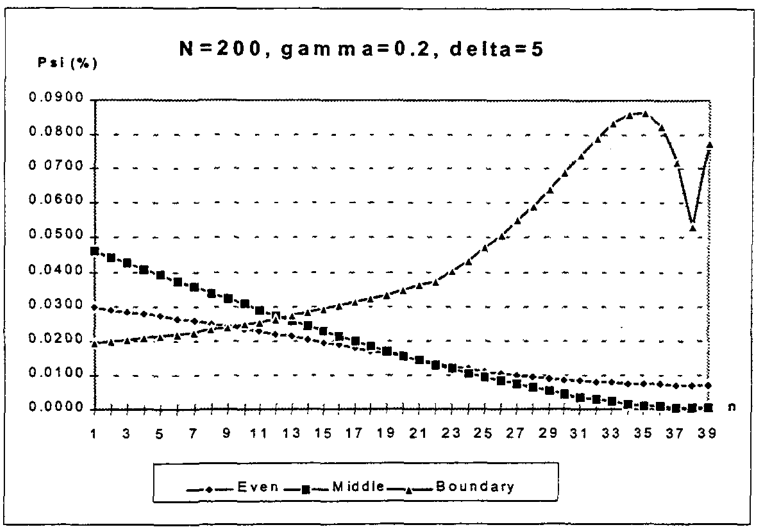




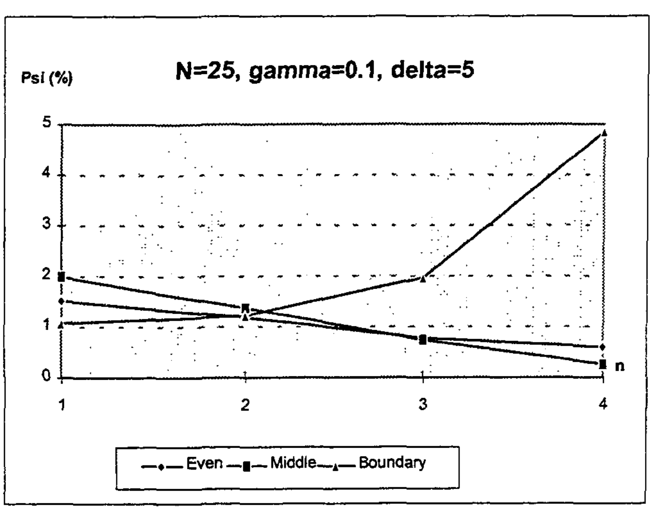


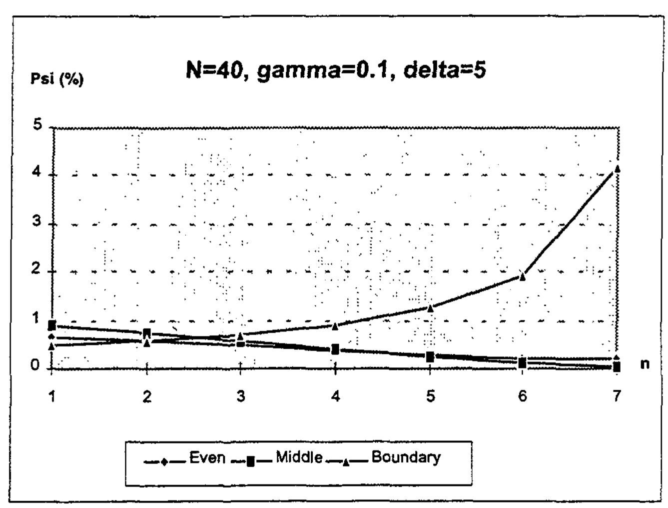


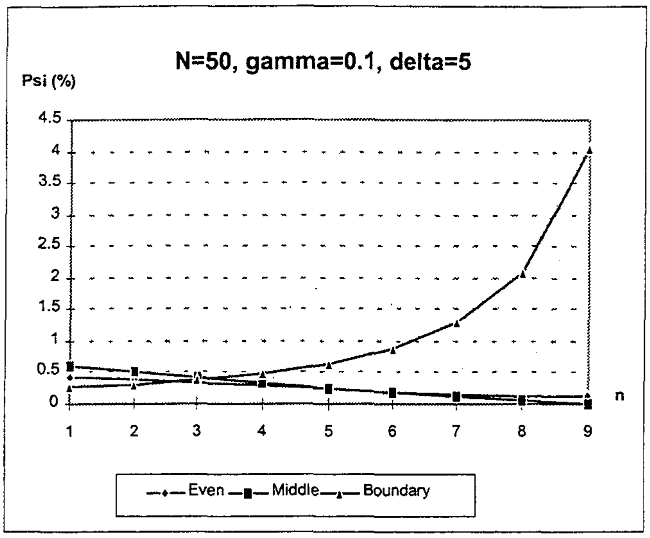


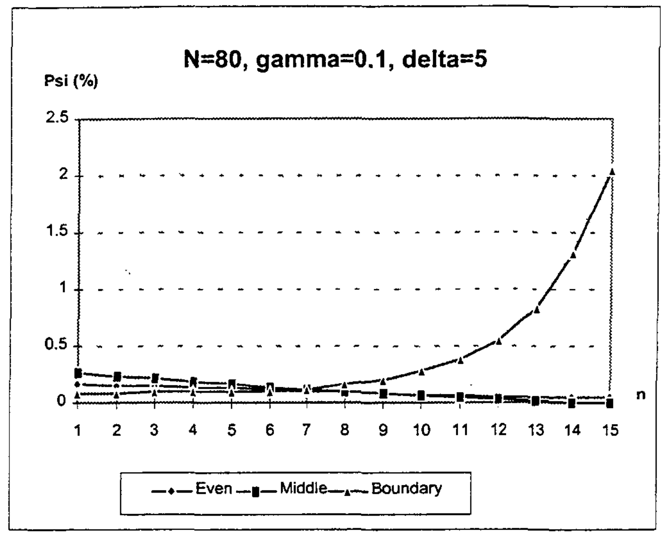


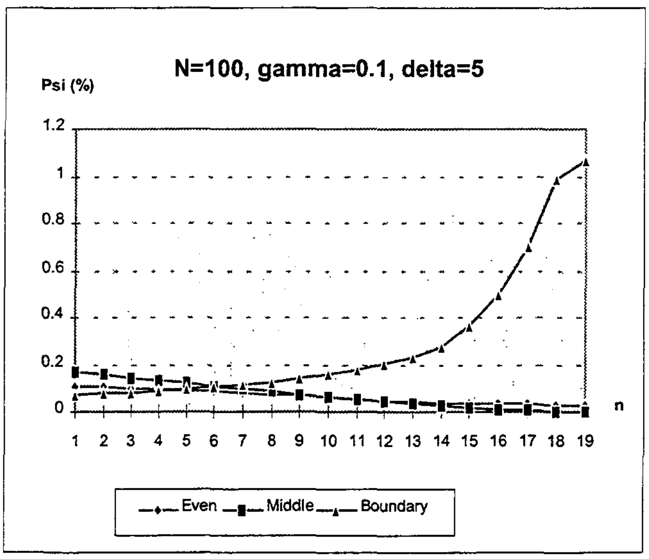
For $s_{max} = \delta = 5$ and $s_{min} = \gamma = 0.1$

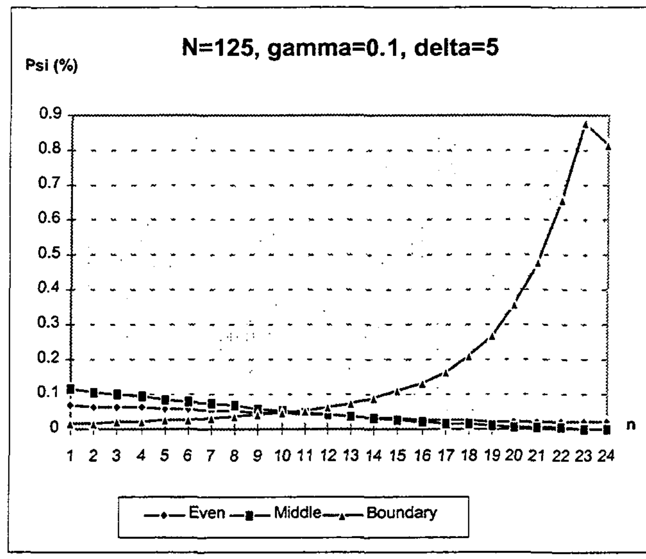


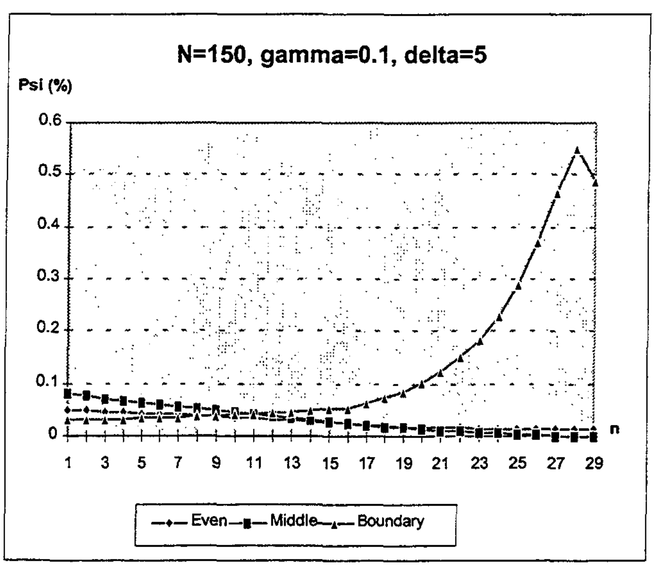


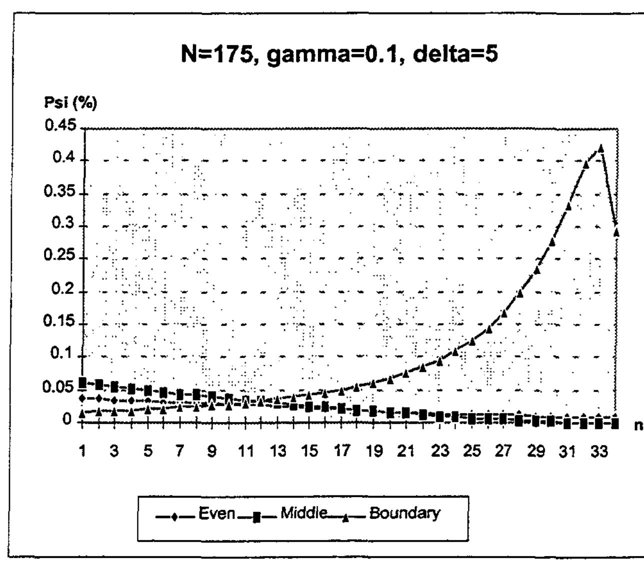

N=50, gamma=0.1, delta=5
Psi (%)
n
Even Middle Boundary

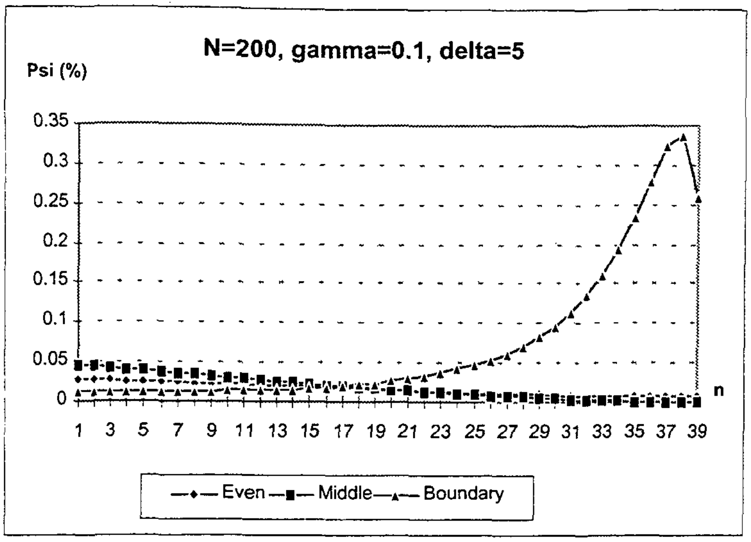

N=80, gamma=0.1, delta=5
Psi (%)
n
Even Middle Boundary


N=100, gamma=0.1, delta=5
Psi (%)
n
Even Middle Boundary

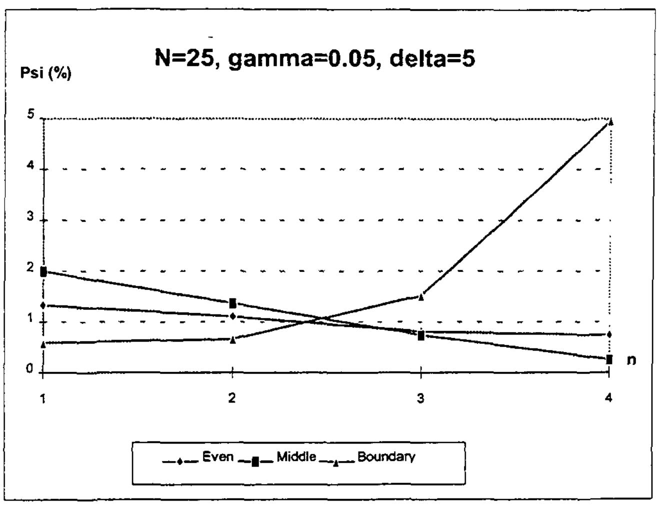

N=125, gamma=0.1, delta=5
Psi (%)
n
Even Middle Boundary

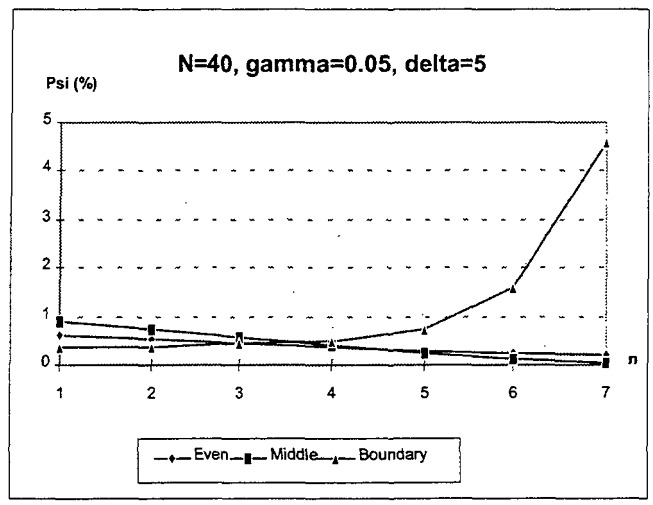

N=150, gamma=0.1, delta=5
Psi (%)
n
Even Middle Boundary

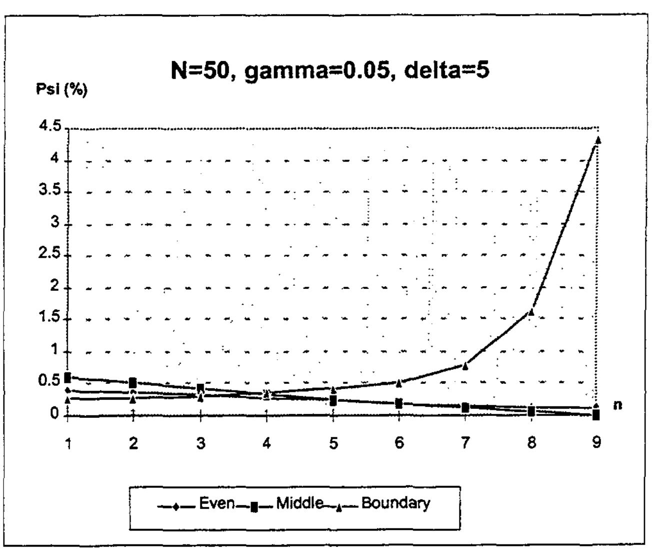

N=175, gamma=0.1, delta=5
Psi (%)
n
Even Middle Boundary

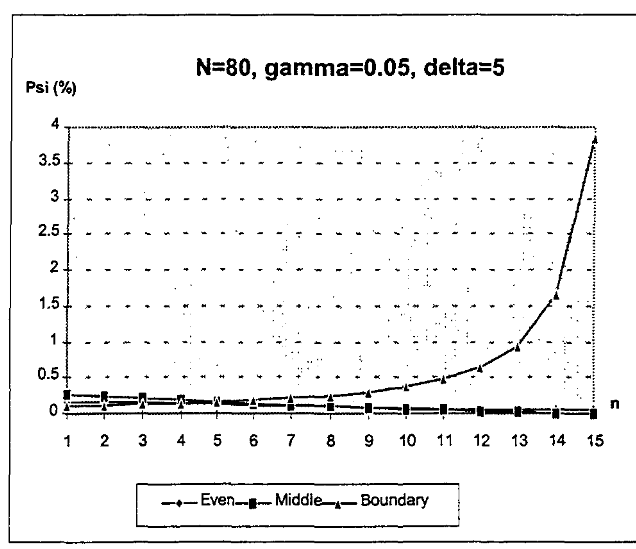


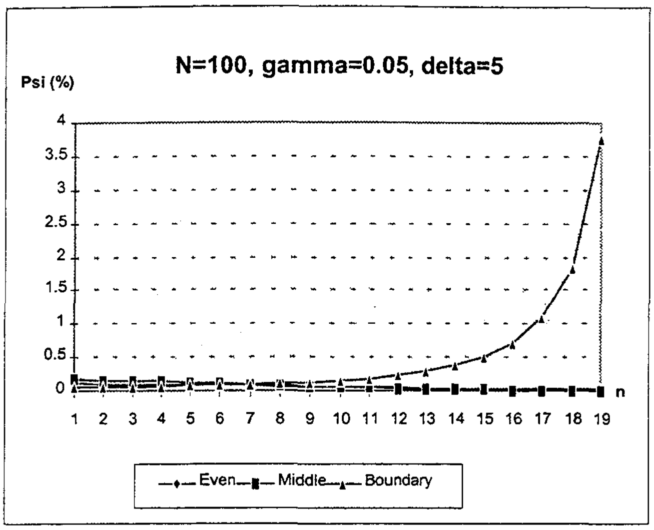
For $s_{max} = \delta = 5$ and $s_{min} = \gamma = 0.05$

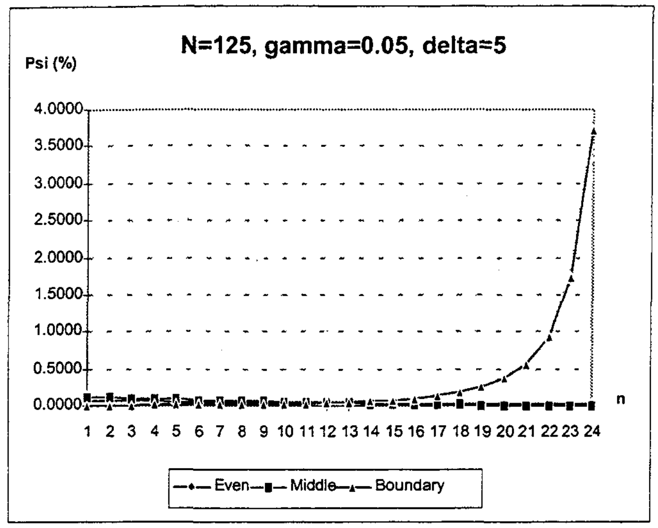


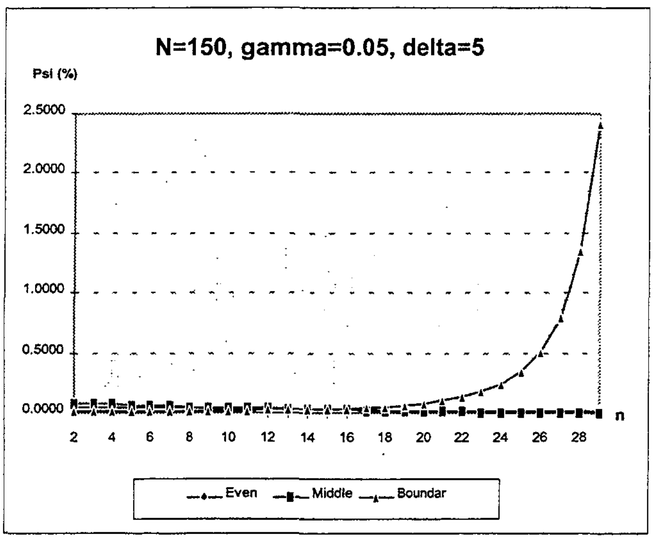


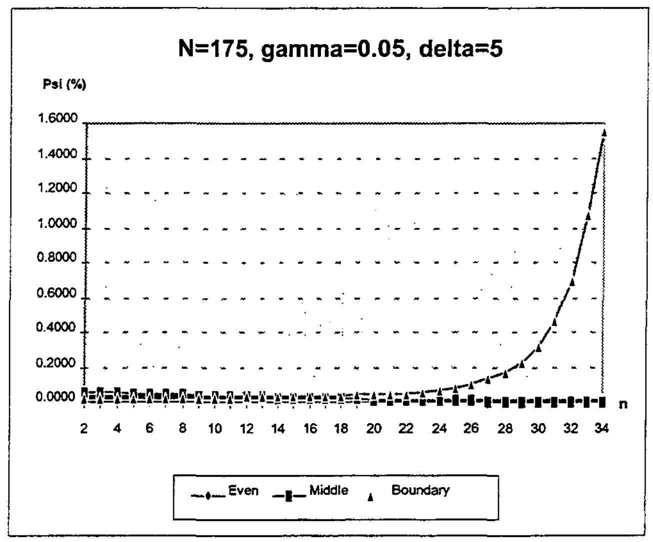


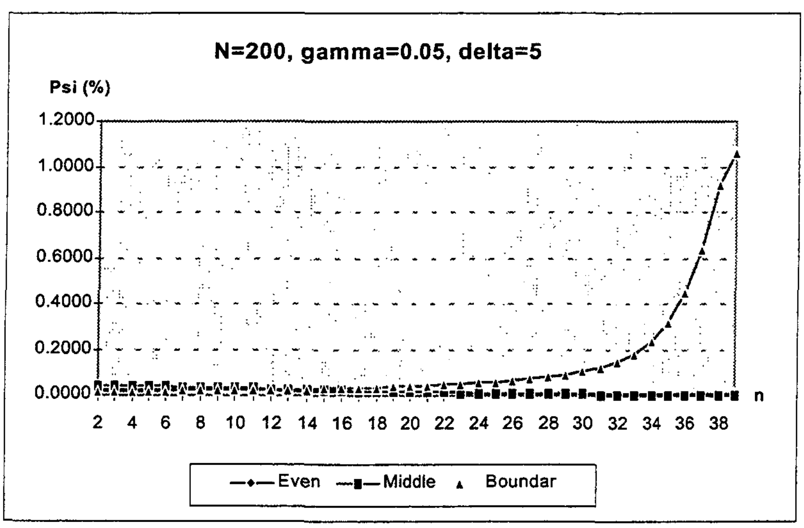



N=100, gamma=0.05, delta=5
Psi (%)
4
3.5
3
2.5
2
1.5
1
0.5
0
n
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
Even
Middle
Boundary

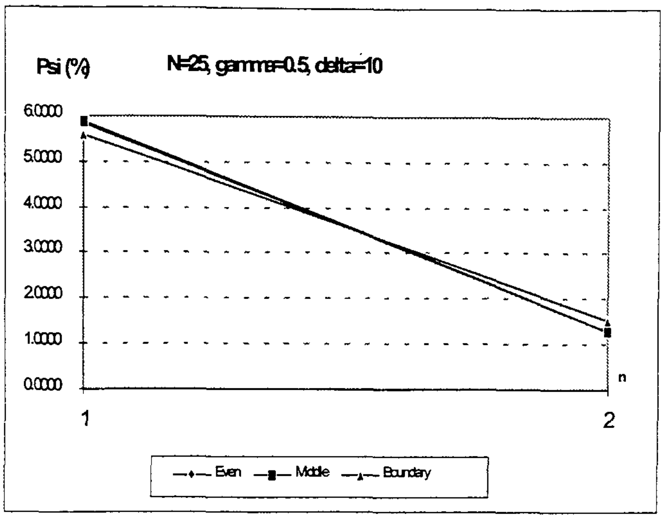

N=125, gamma=0.05, delta=5
Psi (%)
4.0000
3.5000
3.0000
2.5000
2.0000
1.5000
1.0000
0.5000
0.0000
n
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
Even
Middle
Boundary

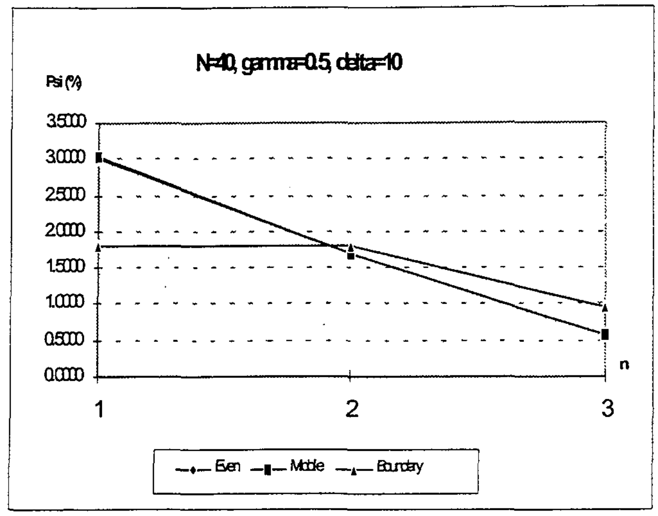

N=150, gamma=0.05, delta=5
Psi (%)
2.5000
2.0000
1.5000
1.0000
0.5000
0.0000
n
2 4 6 8 10 12 14 16 18 20 22 24 26 28
Even
Middle
Boundar

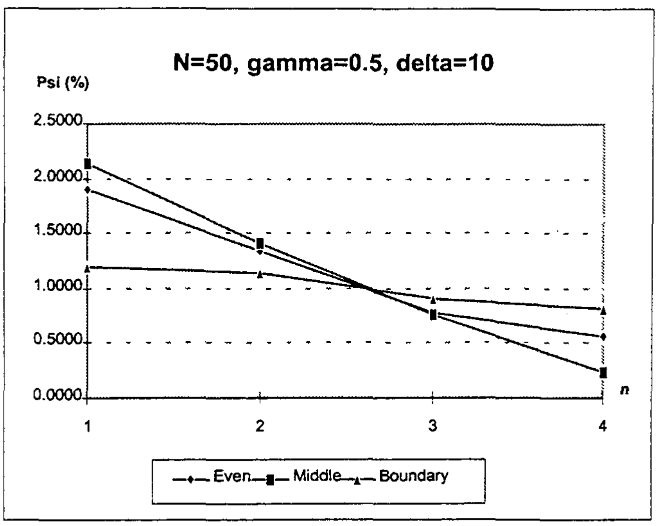

N=175, gamma=0.05, delta=5
Psi (%)
1.6000
1.4000
1.2000
1.0000
0.8000
0.6000
0.4000
0.2000
0.0000
n
2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34
Even
Middle
Boundary

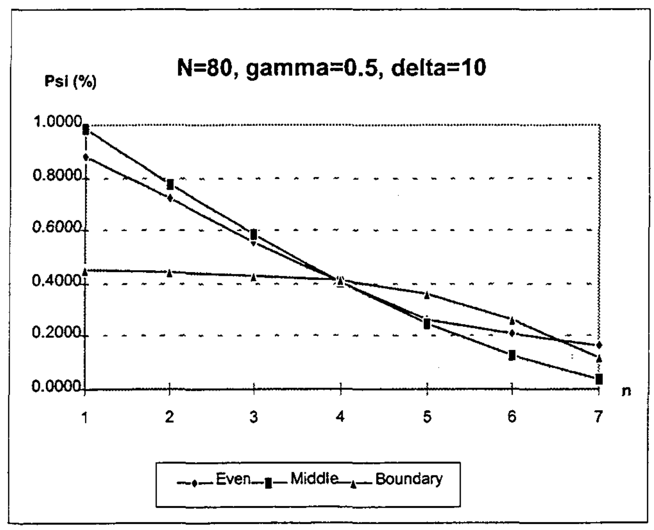

N=200, gamma=0.05, delta=5
Psi (%)
1.2000
1.0000
0.8000
0.6000
0.4000
0.2000
0.0000
n
2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38
Even
Middle
Boundar

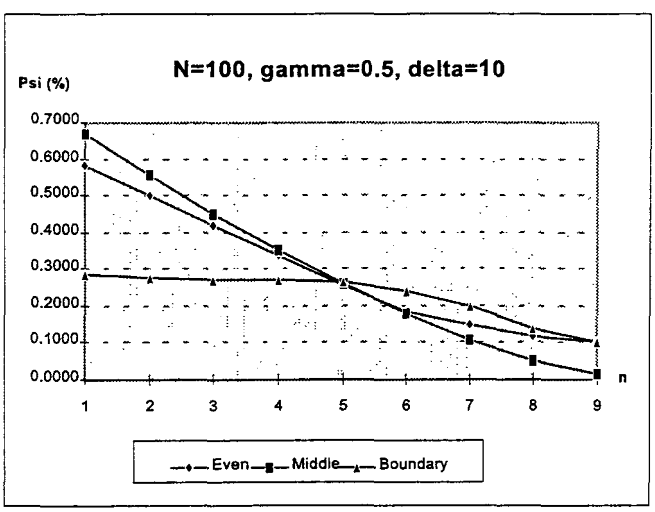
For $s_{max} = \delta = 10$ and $s_{min} = \gamma = 0.5$



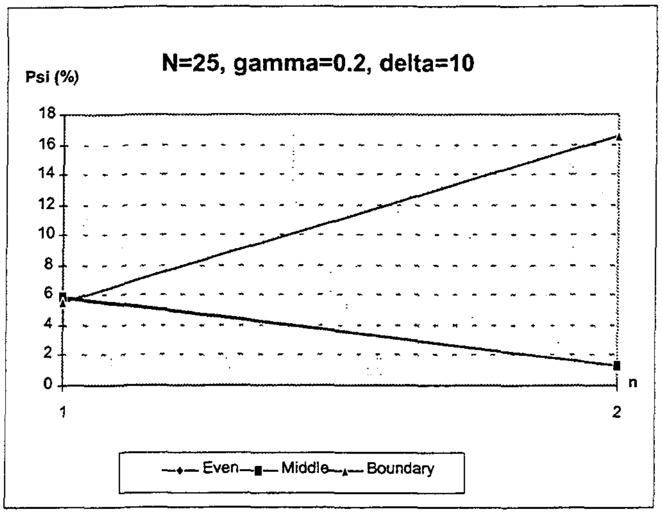


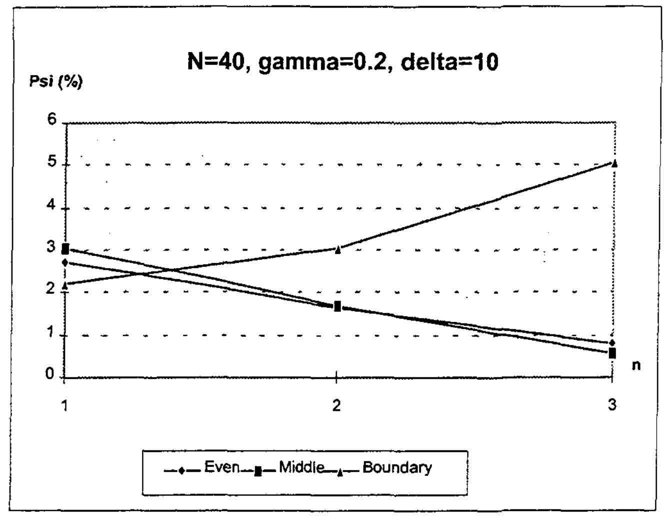


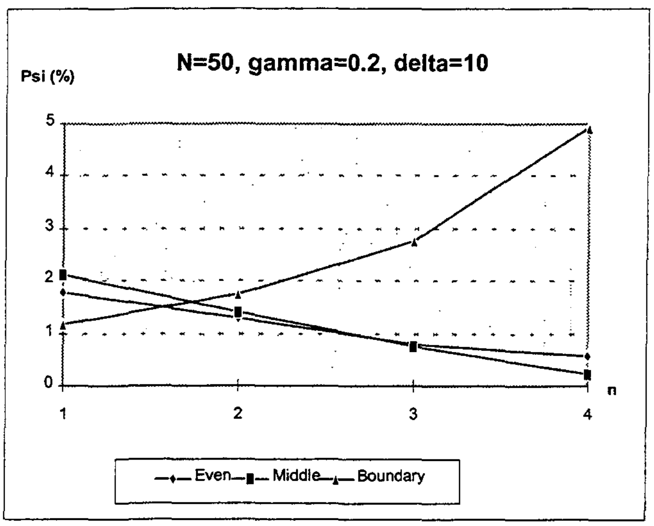


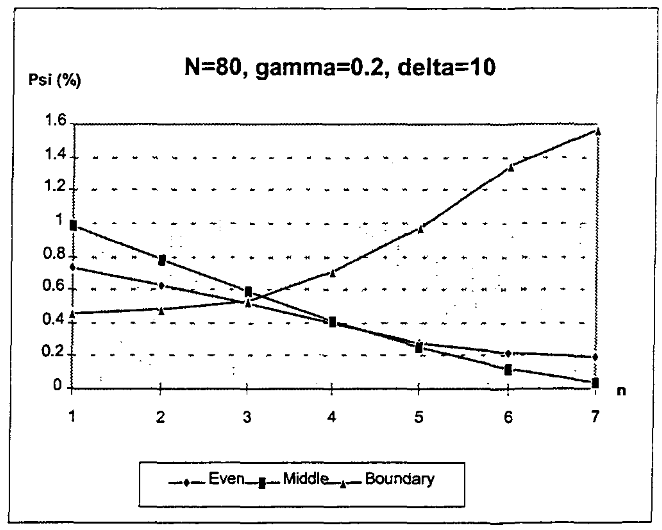


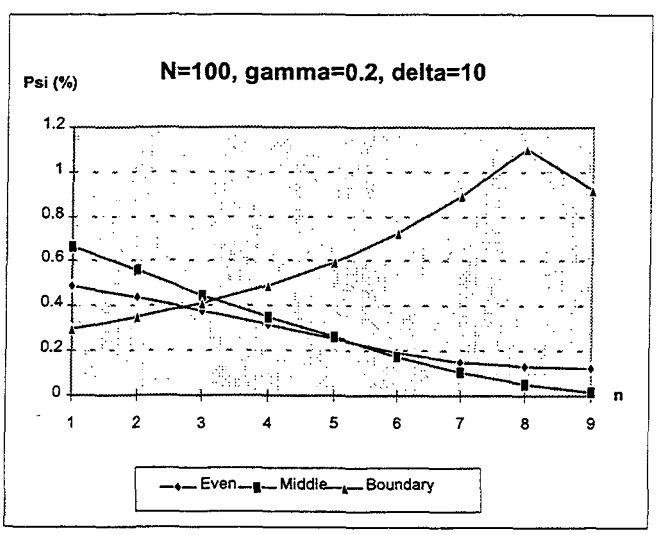
For $s_{max} = \delta = 10$ and $s_{min} = \gamma = 0.2$

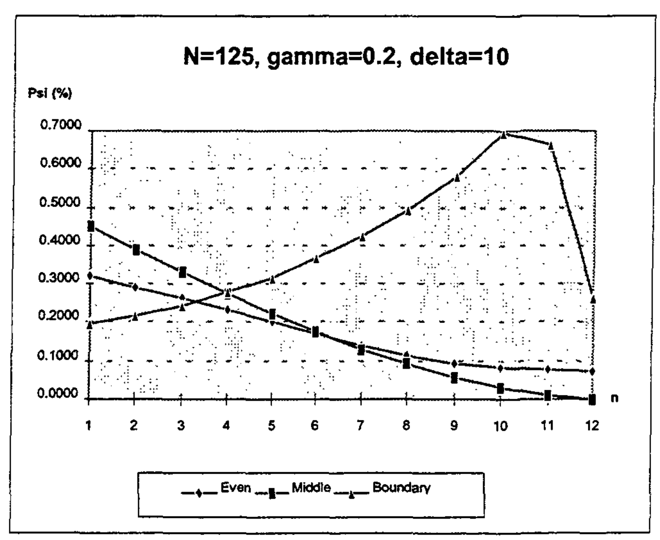


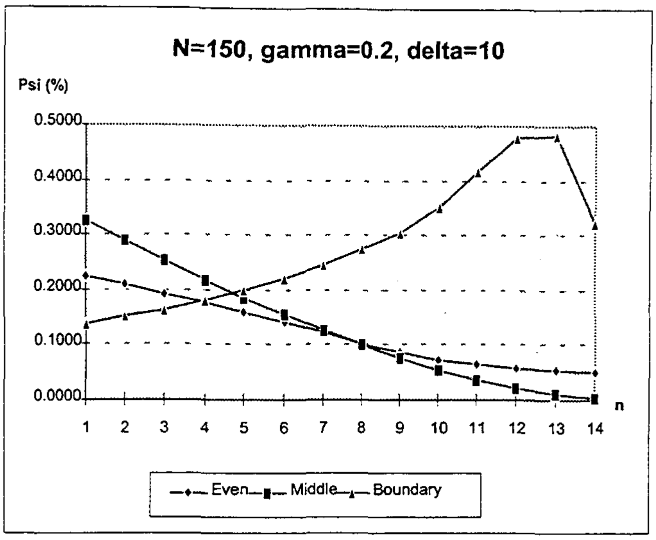


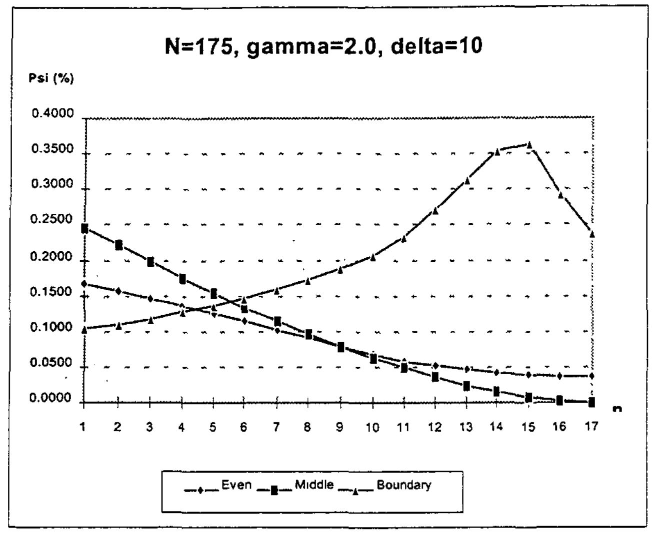


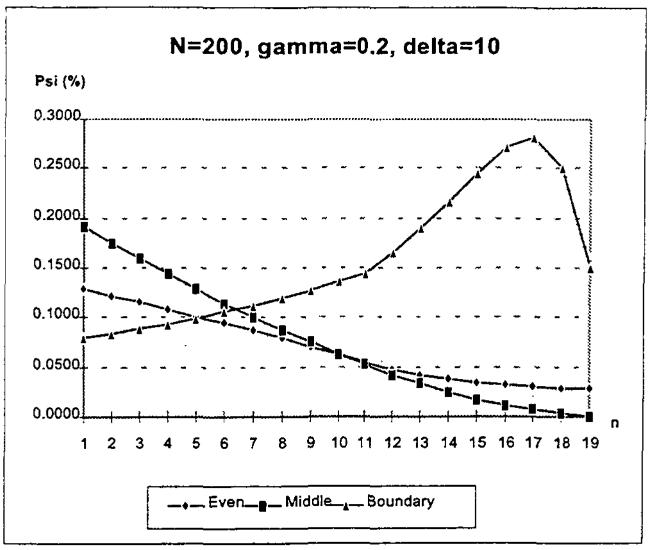




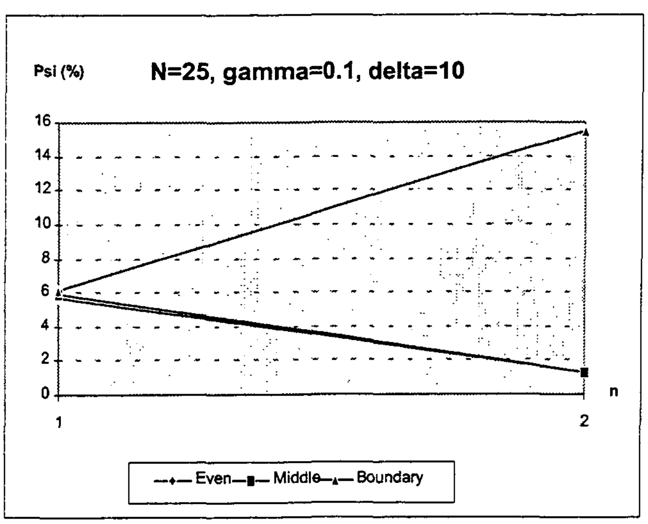


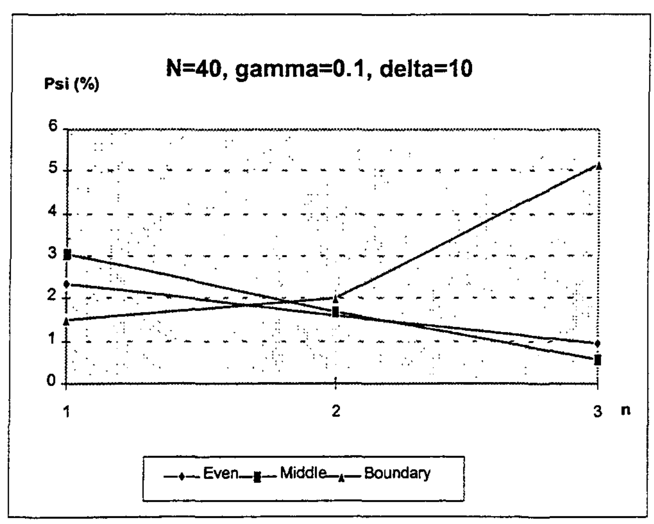


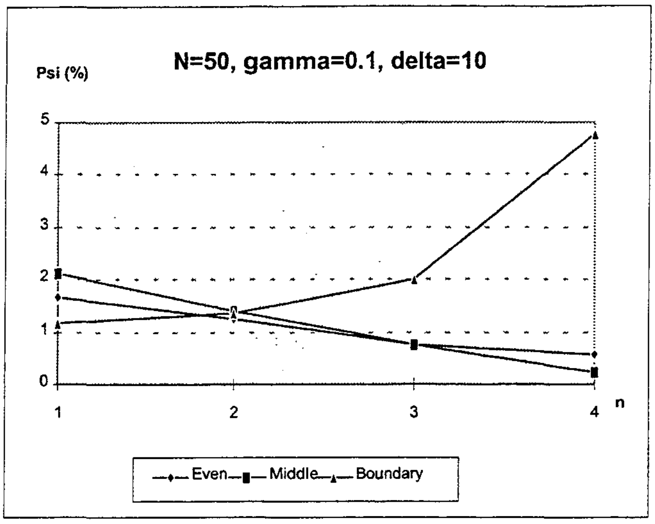


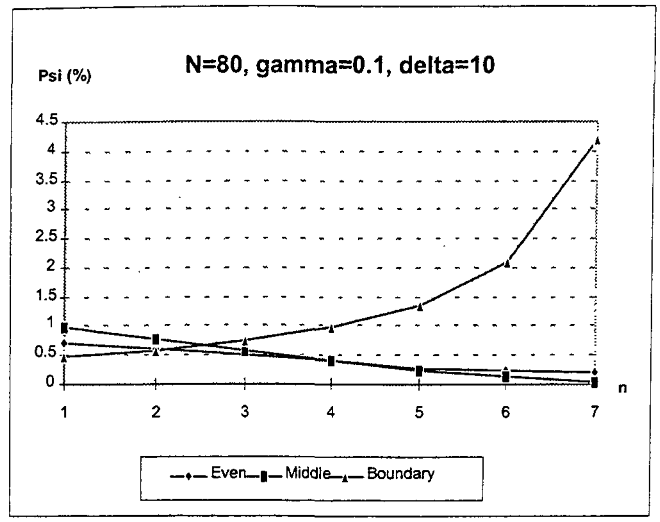


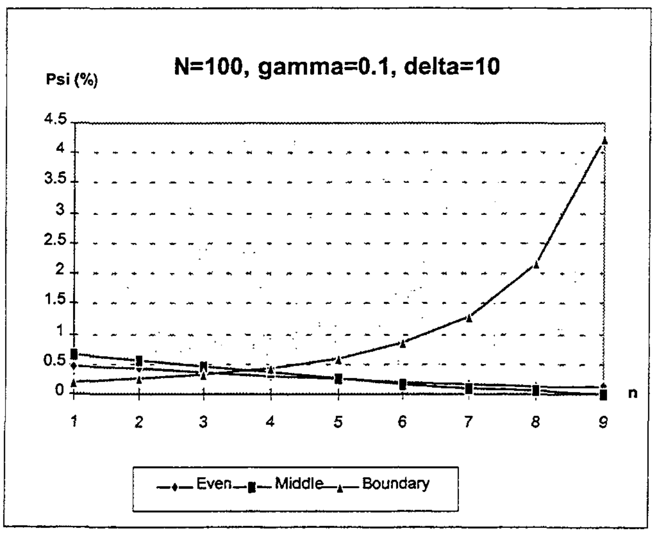
For $s_{max} = \delta = 10$ and $s_{min} = \gamma = 0.1$

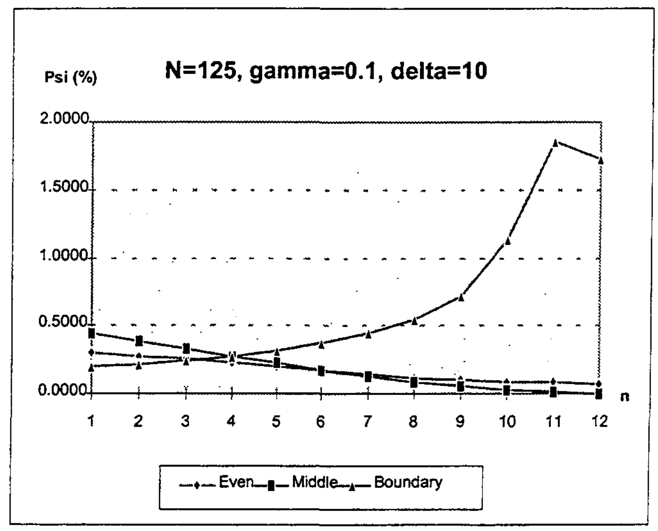


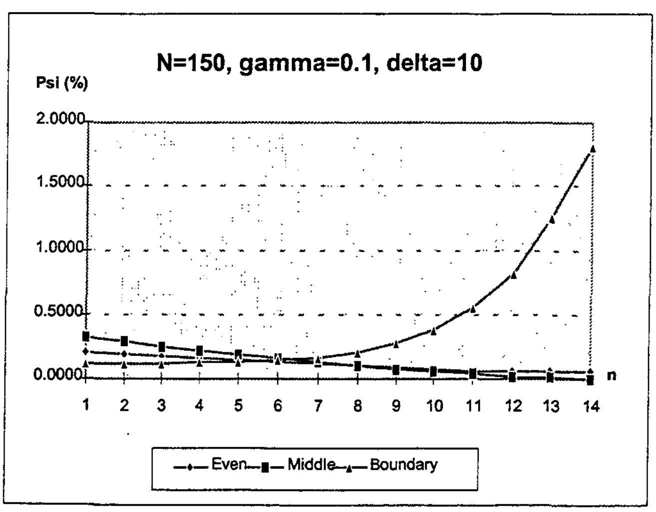


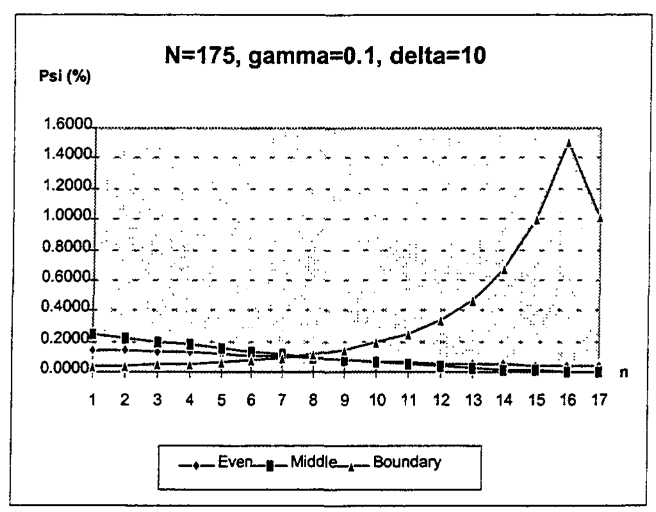

N=50, gamma=0.1, delta=10
Psi (%)
n
Even
Middle
Boundary

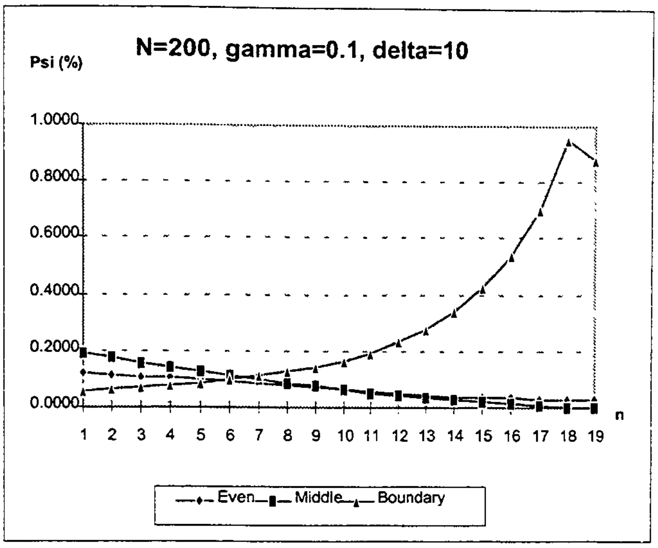

N=80, gamma=0.1, delta=10
Psi (%)
n
Even
Middle
Boundary


N=100, gamma=0.1, delta=10
Psi (%)
n
Even
Middle
Boundary

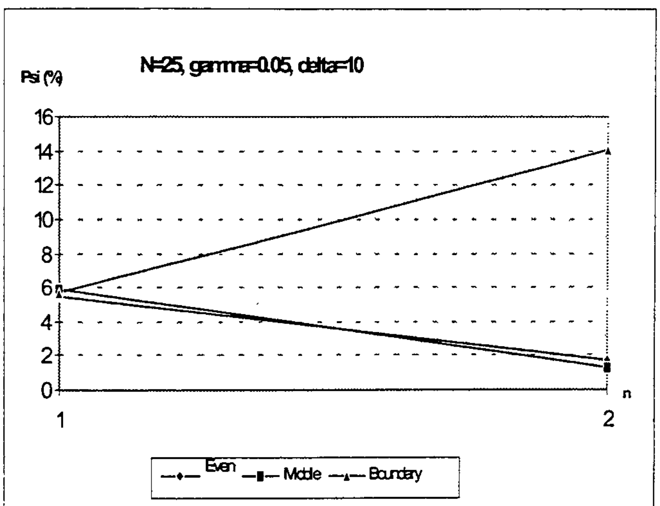

N=125, gamma=0.1, delta=10
Psi (%)
n
Even
Middle
Boundary

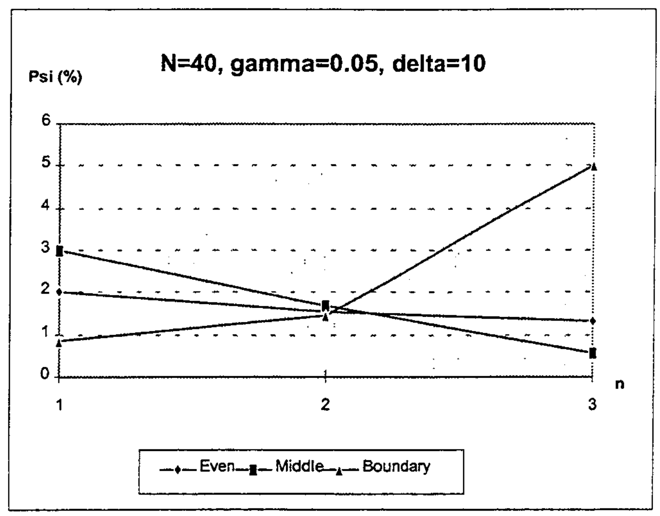

N=150, gamma=0.1, delta=10
Psi (%)
n
Even
Middle
Boundary

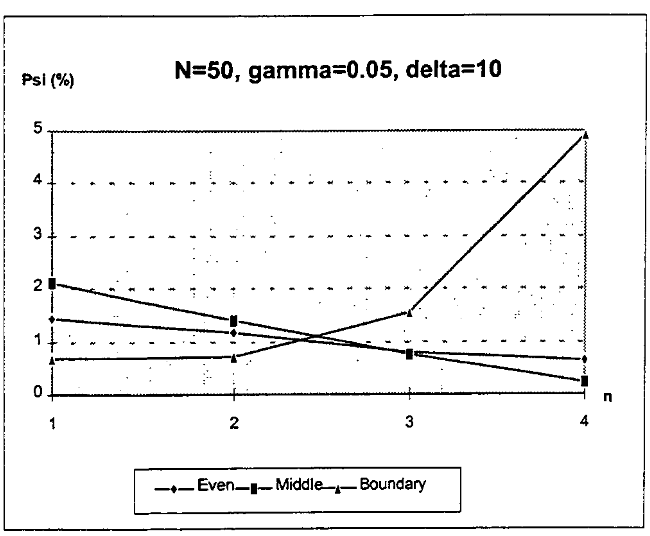

N=175, gamma=0.1, delta=10
Psi (%)
n
Even
Middle
Boundary

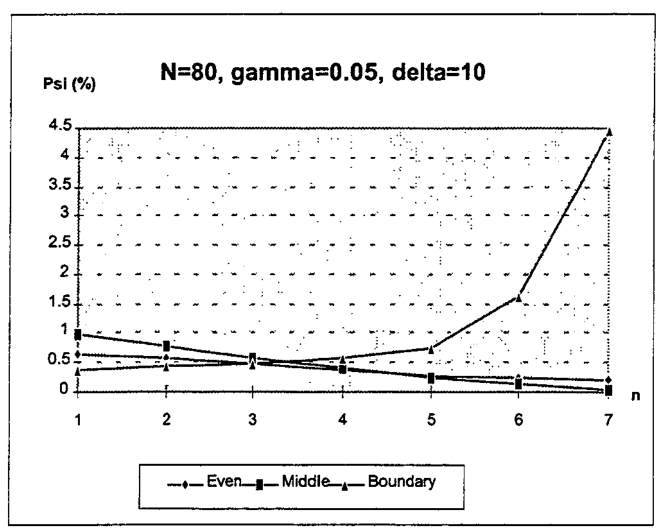


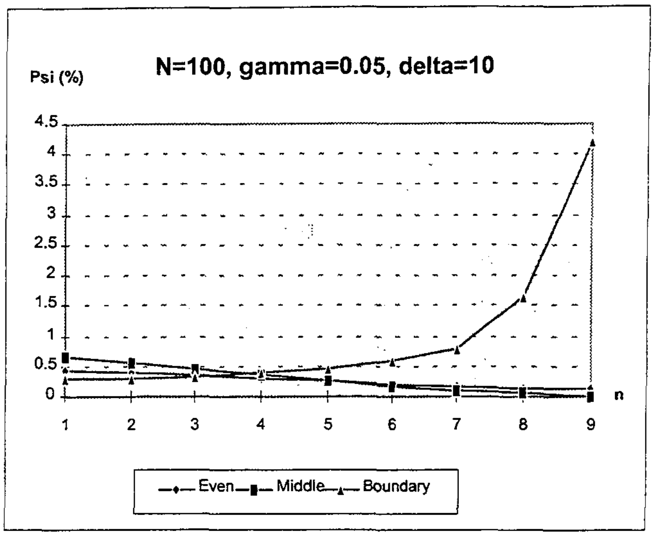
For $s_{max} = \delta = 10$, $s_{min} = \gamma = 0.05$

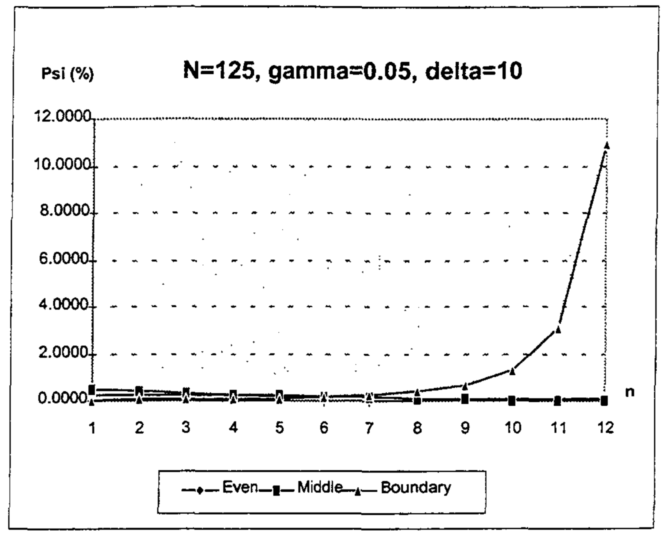


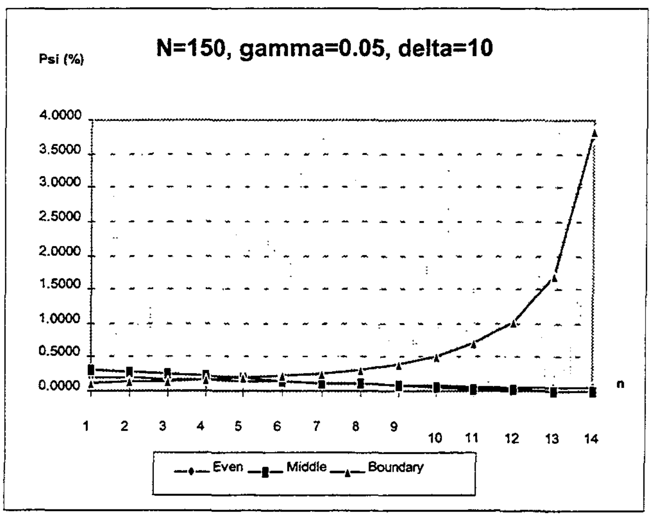


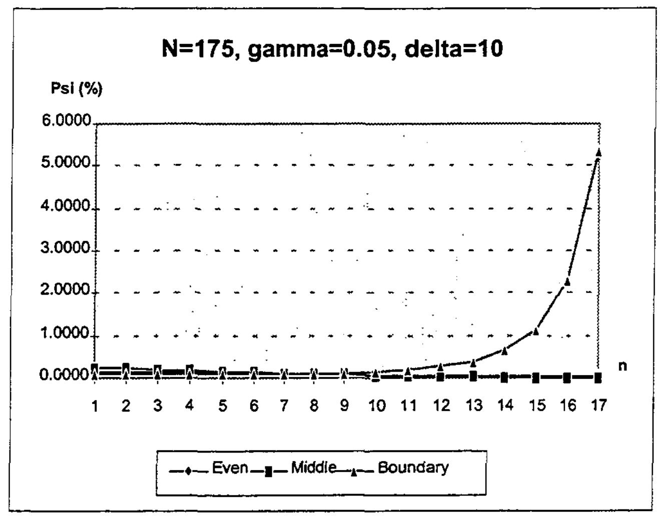


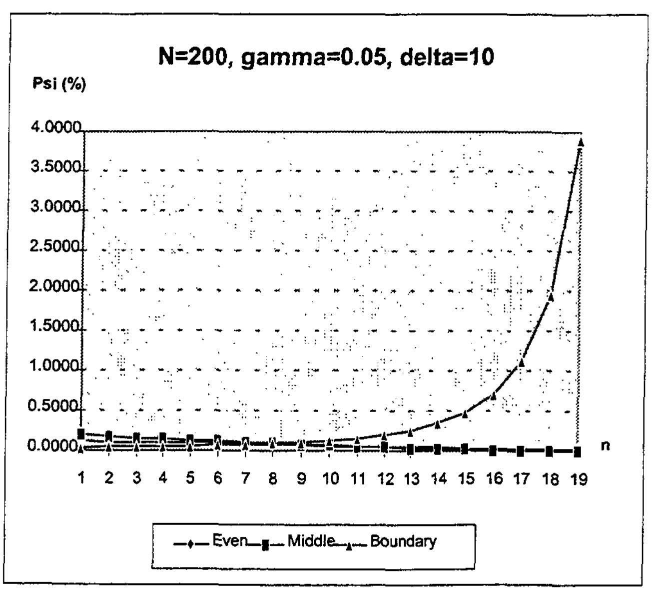



N=100, gamma=0.05, delta=10
Psi (%)
n
Even Middle Boundary


N=125, gamma=0.05, delta=10
Psi (%)
n
Even Middle Boundary


N=150, gamma=0.05, delta=10
Psi (%)
n
Even Middle Boundary


N=175, gamma=0.05, delta=10
Psi (%)
n
Even Middle Boundary


N=200, gamma=0.05, delta=10
Psi (%)
n
Even Middle Boundary

Förteckning över utkomna R&D Reports

R&D Reports är en för IT-enheten och Metodenheten gemensam publikationsserie, som 1988-01-01 ersatte de tidigare "gula" och "gröna" serierna. I serien ingick fram till årsskiftet 1992-93 även **Abstracts** (sammanfattning av metodrapporter från SCB).

Reports published during 1997 and onwards:

1997:1 (grön)	Bortfallsbarometern nr 12 *(Anti Ahtiainen, Stefan Berg, Mats Bergdahl, Fredrik Granström, Dan Hedlin, Lena Otterskog och Monica Rennermalm)*
1997:2 (grön)	Quality Concept for Official Statistics - Entry in the forthcoming update of the Encyclopedia of Statistical Sciences, Wiley & Sons *(Eva Elvers and Bengt Rosén)*
1998:1 (grön)	Preliminär statistik: Nybyggnadskostnader - en simuleringstudie *(Catarina Elffors)*
1998:2 (grön)	On Inclusion Probabilities for Order Sampling *(Bengt Rosén)*
1998:3 (grön)	On the Stratification of Highly Skewed Populations *(Dan Hedlin)*
1998:4 (grön)	Bortfallsbarometer nr 13 *(Per Nilsson, Antti Ahtiainen, Stefan Berg, Mats Bergdahl, Monica Rennermalm och Marcus Vingren)*
1998:5 (grön)	Estimation from Order πps Samples with Non - Response *(Bengt Rosén and Pär Lundqvist)*
1998:6 (grön)	On variance estimation for measures of change when samples are coordinated by a permanent random numbers technique *(Lennart Nordberg)*
1999:1 (grön)	Täckningsproblem i Registret över totalbefolkning RTB. Skattning av övertäckning med en indirekt metod *(Jan Qvist)*
1999:2 (grön)	Bortfallsbarometer nr 14 *(Per Nilsson, Antti Ahtiainen, Mats Bergdahl, Tomas Garås, Jan Qvist och Charlotte Strömstedt)*
1999:3 (grön)	Att mäta statistikens kvalitet *(Claes Andersson, Håkan L. Lindström och Thomas Polfeldt)*
2000:1 (grön)	Kalibrering av vikter – beskrivning av tekniken och de SCB-fall den prövats i *(Sixten Lundström et al)*
2000:2 (grön)	On Inclusion Probabilities and Estimator Bias for Pareto πps Sampling *(Nibia Aires and Bengt Rosén)*

ISSN 0283-8680

Tidigare utgivna **R&D Reports** kan beställas genom Katarina Klingberg, SCB, MET, Box 24 300, 104 51 STOCKHOLM (telefon 08-506 942 82, fax 08-506 945 99, e-post katarina.klingberg@scb.se).
R&D Reports from 1988-1996 can - in case they are still in stock - be ordered from Statistics Sweden, attn. Katarina Klingberg, MET, Box 24 300, SE-104 51 STOCKHOLM (telephone +46 8 506 942 82, fax +46 8 506 945 99, e-mail katarina.klingberg@scb.se).

SCB-Tryck, Örebro 2000.03