Asymptotic Theory for Order Sampling

Bengt Rosén

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Asymptotic Theory for Order Sampling

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Abstract. The paper treats a particular class of sampling schemes, called order sampling schemes, which are executed as follows. Independent random variables, called ordering variables, are associated with the units in the population. A sample of size n is generated by first realizing the ordering variables, and then letting the units with the n smallest ordering variable values constitute the sample.

Although simple to describe and simple to execute, order sampling has an intricate theory. Standard results for linear estimators cannot be applied, since it is impossible to exhibit manageable exact expressions even for first order inclusion probabilities, and matters become only worse for second order quantities. These obstacles are circumvented by asymptotic considerations, which lead to approximation results. The instrumental tool is a general limit theorem for "passage variables". This result prepares for asymptotic results for linear statistic under order sampling. In particular, estimation issues are considered.

By altering the distributions of the ordering variables, a wide class of varying probabilities sampling schemes is obtained. At least two special cases have been considered in the literature; successive sampling and sequential Poisson sampling. These sampling schemes are specified, and their asymptotic theory is studied. Sequential Poisson sampling is, since 1990, used in the Swedish Consumer Price Index survey system.

CONTENTS

Asymptotic Theory for Order Sampling

1 Introduction

1.1 Outline

We consider probability sampling without replacement from a population $U = \{1, 2, ..., N\}$. *Inclusion probabilities* are denoted by $\pi_i = P(I_i = 1)$, $i = 1, 2, ..., N$, where I_i is the *sample inclusion indicator* (= 1 if unit i is sampled, and = 0 otherwise). A *variable* on U, $\mathbf{x} = (x_1, x_2, ..., x_N)$, associates a numerical value to each unit in U. A *linear statistic* is of the form (1.1), where the values of the variable $\mathbf{a} = (a_1, a_2, ..., a_N)$ are presumed to be known for all units in the population, while those of $\mathbf{x} = (x_1, x_2, ..., x_N)$ are known only for sampled units;

$$L(\mathbf{x};\mathbf{a}) = \sum_{i=1}^{N} a_i \cdot x_i \cdot I_i . \tag{1.1}$$

A linear statistic of special interest is the *Horvitz - Thompson (HT) estimator* $\hat{\tau}(\mathbf{x})_{HT}$, which is obtained by setting $a_i = 1/\pi_i$ in (1.1). $\hat{\tau}(\mathbf{x})_{HT}$ yields unbiased estimation of the population total $\tau(\mathbf{x}) = x_1 + x_2 + ... + x_N$. The literature provides formulas, in terms of first and second order inclusion probabilities, for the theoretical variance of $\hat{\tau}(\mathbf{x})_{HT}$ as well as for variance estimators.

This paper treats a particular class of sampling schemes, called *order sampling* schemes, which are executed as follows. With the units in the population are associated independent random variables; ordering variables. A sample of size n is generated by first realizing the ordering variables, and then letting the units with the n *smallest* values constitute the sample.

Although order sampling is simple to describe and simple to execute, its theory is intricate. Standard results for linear estimators cannot be applied, since it is impossible to exhibit manageable expressions even for first order inclusion probabilities, and matters become only worse for second order quantities. To circumvent these obstacles we follow an asymptotic considerations route, which leads to approximation results. The instrumental tool in the considerations is a general limit theorem, which is proved in Section 5. Based on this theorem, asymptotic results for linear statistic under order sampling are formulated in Section 3, and proved in Section 6. In particular, estimation issues are treated in Section 3.3.

By varying the distributions of the ordering variables, a wide class of varying probabilities sampling schemes is obtained. At least two special cases have been considered in the literature; *successive sampling* and *sequential Poisson sampling* (SPS), which correspond to exponential respectively uniform ordering distributions. These sampling schemes are specified in Section 2, and further treated in Section 4. Successive sampling with small sample sizes is considered in most sampling books, see e.g. Cochran (1977), Section 9A.6. Asymptotic results for successive sampling have been studied by i.a. Rosén (1972) and Hájek (1981). Sequential Poisson sampling is, since 1990, used in the Swedish Consumer Price Index survey system, see Ohlsson (1990). Ohlsson (1995a) shows that SPS provides a mean for coordination of pps-samples by permanent random numbers. The investigations which are reported in this paper were initiated by discussions with Dr. Ohlsson on SPS problems. SPS will, however, not be treated in depth here, since a special study of it is presented in Ohlsson (1995b). As regards applications, we presume that other order sampling situations may be encountered in practice, perhaps not as a result of an imposed sampling scheme, but because "nature" sampled in this way.

1.2 Some notation

Here we list some notation which will be used throughout the paper.

Arithmetic operations on (population) variables shall be interpreted component-wise, e.g. for $\mathbf{x} = (x_1, x_2, ..., x_N)$ and $\mathbf{y} = (y_1, y_2, ..., y_N)$, we let $\mathbf{x} \cdot \mathbf{y}$ mean $(x_1 \cdot y_1, ..., x_N \cdot y_N)$.

A standardized version of a general linear statistic is given by the *sample sum* statistic. When the variable is $\mathbf{z} = (z_1, z_2, ..., z_N)$ and the sample size is n, it is denoted by $S(n;\mathbf{z})$;

$$S(n;\mathbf{z}) = \sum_{i=1}^{N} z_i \cdot I_i \ . \tag{1.2}$$

Remark 1.1: The relation

$$L(\mathbf{x};\mathbf{a}) = S(n;\mathbf{a}\cdot\mathbf{x}) \tag{1.3}$$

shows that a linear statistic can be viewed as a sample sum. In particular; $\hat{\tau}(\mathbf{x})_{HT} = S(n;\mathbf{a}/\boldsymbol{\pi})$. With (1.3) as background, most of the following results will be formulated in terms of sample sums, although the main interest concerns more general linear statistics. ♦

Finally some probability notation. P, E and Var denote probability, expected value and variance, and ^{c} stands for centering at expectation, i.e. $Z^c = Z - EZ$. $\xrightarrow{P}$ denotes convergence in probability, and $\Rightarrow$ convergence in distribution (or synonomously in law). $N(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2.

2 Order Sampling

DEFINITION 2.1: ***Order sampling*** with sample size n and *ordering distribution* functions $\mathbf{F} = (F_1, F_2, ..., F_N)$, denoted $OS(n;\mathbf{F})$, is carried out as follows. Independent non-negative random variables $Q_1, Q_2, ..., Q_N$ with absolutely continuous distributions $\mathbf{F}$ on $[0,\infty)$ are associated with the units in the population $U = \{1, 2, ..., N\}$, Q_i with unit i. The Q:s are realized, and the units with the n smallest Q-values constitute the sample. The densities of the ordering distributions are denoted $(f_1, f_2, ..., f_N)$.

Since the ordering distributions are continuous, the realized Q-values contain (with probability 1) no ties. Hence, the sample has fixed size n. If all ordering distributions are equal, $OS(n;\mathbf{F})$ is *simple random sampling*, for which the population units have equal inclusion probabilities. However, as soon as all F_i are not equal, $OS(n;\mathbf{F})$ has non-equal inclusion probabilities. By varying the ordering distributions, $OS(n;\mathbf{F})$ generates quite a wide class of varying probabilities sampling schemes, two of which are specified below.

DEFINITION 2.2: ***a) Sequential Poisson sampling*** is order sampling with uniform ordering distributions. For $\theta = (\theta_1, \theta_2, .., \theta_N)$, $\theta_i > 0$, $SPS(n;\theta)$ stands for $OS(n;\mathbf{F})$ with F_i = the uniform distribution on $[0, 1/\theta_i]$, $i = 1, 2, ..., N$.

b) Successive sampling is order sampling with exponential ordering distributions. For $\theta = (\theta_1, \theta_2, .., \theta_N)$, $\theta_i > 0$, $SUC(n;\theta)$ stands for $OS(n;\mathbf{F})$ with;

$$F_i(t) = 1 - e^{-\theta_i \cdot t}, \text{ which has density } f_i(t) = \theta_i \cdot e^{-\theta_i \cdot t}, \quad 0 \le t < \infty, \quad i = 1, 2, ..., N. \tag{2.1}$$

Remark 2.1: For elaborate discussions of SPS, we refer to Ohlsson (1990, 1995a and b). ♦

Remark 2.2: Definition 2.2.b of successive sampling is not the usual one, which runs as follows. The sample is generated sequentially by n successive draws without replacement. As long as unit i remains in the population U, it is, at each draw, selected among remaining units, with probability proportional to θ_i. The probability for selection of the ordered sample $(i_1, i_2, ..., i_n)$ of different units from U is then, $\tau(\theta) = \theta_1 + \theta_2 + ... + \theta_N$;

$$p(i_1, i_2, \ldots, i_n) = \frac{\theta_{i_1}}{\tau(\theta)} \cdot \frac{\theta_{i_2}}{\tau(\theta) - \theta_{i_1}} \cdot \ \cdots \ \cdot \frac{\theta_{i_n}}{\tau(\theta) - (\theta_{i_1} + \theta_{i_2} + \ldots + \theta_{i_{n-1}})}. \tag{2.2}$$

Next we show that Definition 2.2.b also leads to (2.2), with sample order by increasing Q-values. First some preparations. For $\lambda > 0$, $Exp(\lambda)$ denotes the distribution with density

$\lambda \cdot \exp(-\lambda \cdot t)$, $t \geq 0$. In (i) - (iii) we list well-known properties of independent random variables X and Y with $\text{Exp}(\lambda_X)$ and $\text{Exp}(\lambda_Y)$ distributions respectively. (i) min(X,Y) is $\text{Exp}(\lambda_X + \lambda_Y)$. (ii) $P(X < Y) = \lambda_X/(\lambda_X + \lambda_Y)$. (iii) For a random T which is independent of X and Y, the conditional distribution of X and Y given that $X > T$ and $Y > T$, is the same as the unconditional distribution ("X and Y have no memories").

In the verification of (2.2), we confine to $n = 2$, which suffices to indicate the general principle. Let j and k be different units in U, and let $p(j,k) = P(Q_j$ and Q_k are the two smallest Q-values, and $Q_j < Q_k)$. The following relation (2.3) holds for order sampling in general;

$$p(j,k) = P(Q_j < \min\{Q_i ; i \neq j\}) \cdot P(Q_k < \min\{Q ; i \neq j,k\} | \; Q_i > Q_j, \; i \neq j). \tag{2.3}$$

The variables Q_j and $\min\{Q_i ; i \neq j\}$ are independent. Under (2.1) Q_j is $\text{Exp}(\theta_j)$ and, by (i), $\min\{Q_i ; i \neq j\}$ is $\text{Exp}(\tau(\theta) - \theta_j)$. Hence, by (ii), the left hand factor in (2.3) is $\theta_j/\tau(\theta)$. As regards the right hand factor, first note that, by (iii), it equals the corresponding unconditional probability, then argue as before. The variables Q_k and $\min\{Q_i ; i \neq j,k\}$ are independent, Q_k is $\text{Exp}(\theta_k)$ and, by (i), $\min\{Q_i ; i \neq j,k\}$ is $\text{Exp}(\tau(\theta) - \theta_j - \theta_k)$. Hence, by (ii), $P(Q_k < \min\{Q_i ; i \neq j,k\}) = \theta_k/(\tau(\theta)-\theta_j)$. Thereby relation (2.2) is obtained (for $n = 2$). ♦

3 Asymptotic Distributions for Linear Statistics Under Order Sampling

In this section we consider limit results for linear statistics under order sampling. The practical interest in such results stems from the fact that they lay ground for approximations, which can be employed in practical, "finite" situations. We high-light this aspect by formulating approximation results as well as stringent limit results. The latter provide justification of the former and, moreover, the conditions for the validity of a limit result shed light on the question when a corresponding approximation can be expected to be "good enough". In the first two subsections we confine to the standardized linear statistic, the sample sum. In Section 3.3, which treats estimation issues, we illustrate the statement in Remark 1.1, that results about sample sums readily are transformed to results for general linear statistics.

3.1 The main result in approximation version

APPROXIMATION RESULT 3.1: Consider OS(n; **F**) from U= (1,2,...,N). Let $\mathbf{z} = (z_1, z_2, ..., z_N)$ be a variable on U, and $S(n; \mathbf{z})$ the corresponding sample sum. Then, with μ and σ as specified below, the following holds under general conditions;

$$\text{Law}[S(n;\mathbf{z})] \text{ is well approximated by } N(\mu,\sigma^2). \tag{3.1}$$

Specification of parameters

$$\xi \text{ solves the equation (in t) } \sum_{i=1}^{N} F_i(t) = n, \tag{3.2}$$

$$\mu = \sum_{i=1}^{N} z_i \cdot F_i(\xi), \tag{3.3}$$

$$\phi = \sum_{i=1}^{N} z_i \cdot f_i(\xi) \Big/ \sum_{i=1}^{N} f_i(\xi), \tag{3.4}$$

$$\sigma^2 = \sum_{i=1}^{N} (z_i - \phi)^2 \cdot F_i(\xi) \cdot (1 - F_i(\xi)). \tag{3.5}$$

Remark 3.1: A rephrasing of (3.2) is; $\sum_{i=1}^{N} F_i(\xi) = n$. (3.6)

The sum of distribution functions in (3.2) is non-decreasing, as t increases. The formula (3.4) tacitly presumes that its derivative $\sum_i f_i(\xi)$ is positive in $t=\xi$, which we make to a formal assumption. Then, the solution to (3.2) is *unique*. ♦

3.2 The limit theorem

THEOREM 3.1: For $k=1,2,\ldots$, an $OS(n_k;\mathbf{F}_k)$ sample is drawn from $U_k=(1,2,\ldots,N_k)$. The density for F_{ki} is denoted by f_{ki}. Let $\mathbf{z}_k=(z_{k1},z_{k2},\ldots,z_{kN_k})$ be a variable on U_k and $S_k(n_k;\mathbf{z}_k)$ be the corresponding sample sum. Define ξ_k, μ_k, ϕ_k and σ_k in accordance with (3.2)-(3.5). Then,

$$\text{Law}[\,(S_k(n_k;\mathbf{z}_k)-\mu_k)/\sigma_k] \Rightarrow N(0,1)\,,\ \text{ as } k\to\infty, \tag{3.7}$$

provided that conditions (B1)-(B5) below are satisfied.

(B1) $n_k \to \infty$ as $k \to \infty$.

(B2) $\max_i |z_{ki}-\phi_k|/\sigma_k \to 0$, as $k\to\infty$.

(B3) $0<\varliminf_{k\to\infty} \frac{\xi_k}{n_k}\cdot\sum_{i=1}^{N_k} f_{ki}(\xi_k) \le \varlimsup_{k\to\infty} \frac{\xi_k}{n_k}\cdot\sum_{i=1}^{N_k} f_{ki}(\xi_k)<\infty.$

(B4) For some $\delta>0$, some $C<\infty$ and some function $w(\Delta)$ which tends to 0 as $\Delta\to 0$, the following inequalities hold for $1-\delta\le t,s\le 1+\delta$;

$$|f_{ki}(t\cdot\xi_k)-f_{ki}(s\cdot\xi_k)|\le C\cdot w(|t-s|)\cdot f_{ki}(\xi_k),\ \ i=1,2,\ldots,N_k,\ \ k=1,2,3,\ldots\,. \tag{3.8}$$

(B5) For some $\rho>0$, we have;

$$F_{ki}(\xi_k)\cdot(1-F_{ki}(\xi_k))\ge\rho\cdot\xi_k\cdot f_{ki}(\xi_k),\ \ i=1,2,\ldots,N_k,\ \ k=1,2,3,\ldots\,. \tag{3.9}$$

Proof of the theorem is given in Section 6.

Remark 3.1: The contents of condition (B2) is that no z_{ki}-value is allowed to deviate "extremely" from the majority of z_{ki}-values. (B2) is somewhat involved, though, to the effect that it depends on the ordering distributions as well as on the z_{ki}. A related condition, which pays regard only to the spread of the z_{ki}-values, is *Noether's condition* (N), which is stated below. Let $\bar{z}_k$ and ω_k^2 denote the "ordinary" population mean and population variance;

$$\bar{z}_k=\frac{1}{N_k}\cdot\sum_{j=1}^{N_k} z_{kj} \quad\text{and}\quad \omega_k^2=\frac{1}{N_k-1}\cdot\sum_{i=1}^{N_k}(z_{ki}-\bar{z}_k)^2. \tag{3.10}$$

Introduce the following conditions;

(N) $\max_i |z_{ki}-\bar{z}_k|/(\omega_k\cdot\sqrt{N_k})\to 0$, as $k\to\infty$.

(B6) For some $\delta>0$, $F_{ki}(\xi_k)\cdot(1-F_{ki}(\xi_k))\ge\delta$, $i=1,2,\ldots,N_k$, $k=1,2,3,\ldots$.

(B7) For some $C<\infty$; $|\phi_k-\bar{z}_k|\le C\cdot\omega_k\cdot\sqrt{N_k}$.

Then the following implications hold;

$$(B6)+(N)\Rightarrow(B2). \tag{3.11}$$

$$(B2)+(B7)\Rightarrow(N). \tag{3.12}$$

Verification: For any $\lambda_1,\lambda_2\in[\min_i z_{ki},\max_i z_{ki}]$; $\max_i|z_{ki}-\lambda_1|\le 2\cdot\max_i|z_{ki}-\lambda_2|$. Since ϕ_k and $\bar{z}_k$ both $\in[\min_i z_{ki},\max_i z_{ki}]$ (cf. (3.4) and (3.10)), we have

$$\max_i|z_{ki}-\phi_k|\le 2\cdot\max_i|z_{ki}-\bar{z}_k| \quad\text{and}\quad \max_i|z_{ki}-\bar{z}_k|\le 2\cdot\max_i|z_{ki}-\phi_k|. \tag{3.13}$$

Presume that (B6) is satisfied. By (3.5), (B6) and the fact that a sum of squared deviations is minimal around the mean, we have;

$$\sigma_k^2 = \sum_{i=1}^{N_k} (z_{ki} - \phi_k)^2 \cdot F_{ki}(\xi_k) \cdot (1 - F_{ki}(\xi_k)) \geq \rho \cdot \sum_{i=1}^{N_k} (z_{ki} - \phi_k)^2 \geq$$

$$\geq \rho \cdot \sum_{i=1}^{N_k} (z_{ki} - \bar{z}_k)^2 = \delta \cdot (N_k - 1) \cdot \omega_k^2. \tag{3.14}$$

From (3.14) and (3.13) it is seen that (3.11) holds. To verify (3.12);

$$\sigma_k^2 = \sum_{i=1}^{N_k} (z_{ki} - \phi_k)^2 \cdot F_{ki}(\xi_k) \cdot (1 - F_{ki}(\xi_k)) \leq \sum_{i=1}^{N_k} (z_{ki} - \phi_k)^2 =$$

$$= \sum_{i=1}^{N_k} (z_{ki} - \bar{z}_k)^2 + N_k \cdot (\phi_k - \bar{z}_k)^2 = (N_k - 1) \cdot \omega_k^2 + N_k \cdot (\phi_k - \bar{z}_k)^2. \tag{3.15}$$

The claim in (3.12) now follows from (3.15), (3.13) and (B7). ♦

Remark 3.2: Condition (B8) below may be viewed as an implicit part of condition (B1);

(B8) $N_k \to \infty$, as $k \to \infty$.

Also the following implication holds. The straightforward proof is omitted.

Any of (B2) or (N) $\Rightarrow$ (B6). (3.16)

3.3 Estimation of population characteristics from an order sample

The central inference problem in finite population sampling concerns estimation of a population total $\tau(\mathbf{x}) = x_1 + x_2 + ... + x_N$. The results *in Sections* 3.1 *and* 3.2 *contribute to this estima*tion problem, as stated below.

APPROXIMATION RESULT 3.2: Let assumptions and notation be as in Approximation Result 3.1. Consider the following linear statistic;

$$\hat{\tau}(x)_{OS} = \sum_{i=1}^{N} \frac{x_i}{F_i(\xi)} \cdot I_i. \tag{3.17}$$

Then a) and b) below hold under general conditions;

a) With χ^2 as stated below,

Law[$\hat{\tau}(\mathbf{x})_{OS}$] is well approximated by N($\tau(\mathbf{x}), \chi^2$). (3.18)

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{x_i}{F_i(\xi)} - \gamma \right)^2 \cdot F_i(\xi) \cdot (1 - F_i(\xi)), \tag{3.19}$$

where

$$\gamma = \sum_{i=1}^{N} \frac{x_i}{F_i(\xi)} \cdot f_i(\xi) \Big/ \sum_{i=1}^{N} f_i(\xi). \tag{3.20}$$

b) A variance estimator for $\hat{\tau}(\mathbf{x})_{OS}$ is provided by;

$$\hat{V}[\hat{\tau}(x)_{OS}] = \frac{n}{n-1} \cdot \sum_{i=1}^{N} \left(\frac{x_i}{F_i(\xi)} - \hat{\gamma} \right)^2 \cdot (1 - F_i(\xi)) \cdot I_i, \tag{3.21}$$

where

$$\hat{\gamma} = \sum_{i=1}^{N} \frac{x_i}{F_i(\xi)^2} \cdot f_i(\xi) \cdot I_i \Big/ \sum_{i=1}^{N} \frac{f_i(\xi)}{F_i(\xi)} \cdot I_i. \tag{3.22}$$

Justification: The a)-part is obtained by setting $z_i = x_i / F_i(\xi)$ in Approximation Result 3.1. For the b)-part, we first note that the limit version of (3.18) implies that $\hat{\tau}(\mathbf{x})_{OS}$ is an asymptotically unbiased estimator of $\tau(\mathbf{x})$. By combining this with the fact that the HT-estimator $\hat{\tau}(\mathbf{x})_{HT}$ is unique as unbiased linear estimator of $\tau(\mathbf{x})$, one is led to the conjecture that good approximations of OS(n; **F**) inclusion probabilities, under general conditions are given by;

$$\pi_i \approx F_i(\xi), \quad i = 1, 2, ..., N. \tag{3.23}$$

The justification of the variance estimator (3.21) is heuristic to the effect that it relies on (3.23) being a good approximation. In the first round, disregard the factor n/(n-1) in (3.21) and replace $\hat{\gamma}$ by γ. By taking expectation in (3.21), using (3.23) as an exact relation, and comparing the result with (3.19), it is seen that $\hat{V}(\hat{\tau}(\mathbf{x}))$ yields (formally) unbiased estimation of χ^2. However, in a practical situation γ is not known, but has to be estimated. By using (3.23) as an exact relation again, it is seen that the denominator and nominator in (3.22) yield (formally) unbiased estimation of the corresponding quantities in γ in (3.20). Hence, $\hat{\gamma}$ is a natural estimator of γ. The factor n/(n-1), finally, is inserted on the following ground. In order that (3.21) should yield the standard SRS variance estimator when all F_i are equal, the adjustment factor n/(n-1) is needed. Based on simulations, Rosén (1992), we know that the adjustment works well in many successive sampling situations, and we conjecture that it is good also for general order sampling. ♦

4 Asymptotic Results for Sequential Poisson and Successive Sampling

Here we specialize the results in Sections 3.1 and 3.2 to sequential Poisson and successive sampling (see Definition 2.2. The parameter $\theta = (\theta_1, \theta_2, .., \theta_N)$ is said to be *normalized* if $\theta_1 + \theta_2 + .. + \theta_N = 1$.

4.1 Sequential Poisson sampling

Since a study of SPS is presented in Ohlsson (1995b), we are brief on this topic, and we confine to specializing Approximation Result 3.1.

APPROXIMATION RESULT 4.1: Consider SPS(n; θ) from U = (1, 2, ..., N), with normalized θ. Assume that $n \cdot \theta_i < 1$, $i = 1, 2, ..., N$. Let **z** be a variable on U and S(n; **z**) the corresponding sample sum. Then (3.7) holds with the following μ and σ;

$$\mu = n \cdot \sum_{i=1}^{N} z_i \cdot \theta_i , \tag{4.1}$$

$$\sigma^2 = \sum_{i=1}^{N} \left(z_i - \sum_{j=1}^{N} z_j \cdot \theta_j \right)^2 \cdot n\theta_i \cdot (1 - n\theta_i) . \tag{4.2}$$

Derivation from Approximation Result 3.1: By Definition 2.2, SPS(n; θ) is OS(n; **F**) with $F_i(t) = \min(t \cdot \theta_i, 1)$, and $f_i(t) = \theta_i$ on $t \in [0, 1/\theta_i]$ and $= 0$ outside $[0, 1/\theta_i]$, $0 \le t < \infty$, $i = 1, 2, ..., N$. On the interval $0 < t < 1/\max_i \theta_i$, all $f_i(t)$ are continuous, and (3.2) takes the form $t \cdot (\theta_1 + \theta_2 + .. + \theta_N) = 1$, which under normalized θ yields $\xi = n$. When $n \cdot \theta_i < 1$, $i = 1, 2, ..., N$, $\xi = n$ in fact lies on the interval $(0, 1/\max_i \theta_i)$. It is readily checked that (3.3) and (3.5) here specialize to (4.1) and (4.2). ♦

4.2 Successive sampling

Here we specialize the limit theorem as well as the approximation result.

APPROXIMATION RESULT 4.2: Consider SUC(n; θ) from U = (1, 2,...,N), with normalized θ. Let **z** be a variable on U and S(n; **z**) the corresponding sample sum. Then (3.7) holds with μ and σ as stated below;

ξ is the solution to the equation (in t) $\sum_{i=1}^{N}(1-e^{-\theta_i \cdot t}) = n.$ (4.3)

$$\mu = \sum_{i=1}^{N} z_i \cdot (1-e^{-\theta_i \cdot \xi}), \tag{4.4}$$

$$\phi = \frac{\sum_{i=1}^{N} z_i \cdot \theta_i \cdot e^{-\theta_i \cdot \xi}}{\sum_{i=1}^{N} \theta_i \cdot e^{-\theta_i \cdot \xi}}, \tag{4.5}$$

$$\sigma^2 = \sum_{i=1}^{N}\left(z_i - \frac{\sum_{j=1}^{N} z_j \cdot \theta_j \cdot e^{-\theta_j \cdot \xi}}{\sum_{j=1}^{N} \theta_j \cdot e^{-\theta_j \cdot \xi}} \right)^2 \cdot (1-e^{-\theta_i \cdot \xi}) \cdot e^{-\theta_i \cdot \xi}. \tag{4.6}$$

Derivation from Approximation Result 3.1: It is readily checked that (3.2) - (3.5) specialize to (4.3) - (4.6) when, see (2.1), $F_i(t) = 1-e^{-\theta_i \cdot t}$ and $f_i(t) = \theta_i \cdot e^{-\theta_i \cdot t}, 0 \le t < \infty$. ♦

THEOREM 4.1: For $k=1,2,3,\ldots$, consider $SUC(n_k;\theta_k)$ from $U_k=(1,2,\ldots,N_k)$, with normalized θ_k. Let $\mathbf{z}_k$ be a variable on U_k and $S_k(n_k;\mathbf{z}_k)$ the corresponding sample sum. Let ξ_k, μ_k, ϕ_k and σ_k be in accordance with (4.3) - (4.6). Then (3.7) holds, if conditions (C1)-(C4) below are satisfied.

(C1) $n_k \to \infty$ as $k \to \infty$.

(C2) $\max_i |z_{ki} - \phi_k| / \sigma_k \to 0$, as $k \to \infty$.

(C3) For some $L < \infty$; $\theta_{ki} \le L/n_k$, $i=1,2,\ldots,N_k$, $k=1,2,3,\ldots$.

(C4) For some $\rho>0$, $\gamma>1$ and $\delta>0$, the following holds;

$$\#\{i: \theta_{ki} \ge \rho/N_k\} \ge \max(\gamma \cdot n_k, \delta \cdot N_k\}, \quad k=1,2,3,\ldots .$$

Remark 4.1: The above theorem improves results in Rosén (1972) and Hájek (1981), to the effect that the conditions for asymptotic normality are more general. ♦

Remark 4.2: Here we comment on condition (C4). When the θ:s are normalized, the average θ_{ki}-value is $1/N_k$. Say that θ_{ki} is "small" if $\theta_{ki} < \rho/N_k$ (for some "small" ρ), otherwise say that it is "non-small". (C4) requires that the number of non-small θ is at least a bit larger than the sample size and, moreover, that the non-small θ comprise a non-negligible proportion of all θ.

Below we list some conditions which relate to (C4);

(C5) For some $\rho > 0$; $\theta_{ki} \ge \rho/N_k$, $i=1,2,\ldots,N_k$, $k=1,2,3,\ldots$.

(C6) For some $\lambda < 1$; $n_k \le \lambda \cdot N_k$, $k=1,2,3,\ldots$.

The following implications hold;

(C4) $\Rightarrow$ (C6), (4.7)

(C5) + (C6) $\Rightarrow$ (C4). (4.8)

To realize (4.7), presume that (C4) holds. Then, $N_k \ge \#\{i: \theta_{ki} \ge \rho/N_k\} \ge \max(\gamma \cdot n_k, \delta \cdot N_k\} \ge \gamma \cdot n_k$, from which (4.7) follows. To realize (4.8), presume that (C5) and (C6) hold. Then, $\#\{i: \theta_{ki} \ge \rho/N_k\} = N_k \ge$ (by (C6)) $\ge \max(\lambda^{-1} \cdot n_k, N_k\}$. Hence (C4) is satisfied. ♦

Proof of Theorem 4.1: We shall show that (C1) - (C4) imply (B1) - (B5) in Theorem 3.1, which then yields Theorem 4.1. Set,

$$\varphi_k(t) = \frac{1}{n_k} \cdot \sum_{i=1}^{N_k} F_{ki}(t) = \frac{1}{n_k} \cdot \sum_{i=1}^{N_k} (1 - e^{-\theta_{ki} \cdot t}), \quad 0 \le t < \infty. \tag{4.9}$$

To estimate φ_k from above, use the inequality $1 - e^{-x} \le x,\ 0 \le x < \infty$;

$$\varphi_k(t) \le \frac{1}{n_k} \cdot \sum_{i=1}^{N_k} \theta_{ki} \cdot t = (\text{since } \theta_k \text{ is normalized}) = \frac{t}{n_k} =: \varphi_k^{(u)}(t), \quad 0 \le t < \infty. \tag{4.10}$$

Next we estimate φ_k from below. Set $A_k = \#\{i: \theta_{ki} \ge \rho/N_k\}$. By (4.9) and (C4) we have;

$$\varphi_k(t) \ge \frac{A_k}{n_k} \cdot (1 - e^{-\rho \cdot t/N_k}) =: \varphi_k^{(\ell)}(t), \quad 0 \le t < \infty. \tag{4.11}$$

Since $\varphi_k^{(\ell)}(t) \le \varphi_k(t) \le \varphi_k^{(u)}(t)$, and ξ_k is the solution to $\varphi_k(t) = 1$, ξ_k lies between the solutions to $\varphi_k^{(u)}(t) = 1$ and $\varphi_k^{(\ell)}(t) = 1$, which yields (4.12) below, where we also use $\ln(1-x)^{-1} \le x/(1-x)$.

$$n_k \le \xi_k \le \frac{N_k}{\rho} \cdot \ln(1 - n_k/A_k)^{-1} \le (\text{by (C4)}) \le \frac{n_k}{\rho \cdot \delta \cdot (1-\gamma^{-1})}, \quad k = 1,2,3,\ldots. \tag{4.12}$$

As a consequence of (C3) and (4.12), we have, for some $C < \infty$;

$$\xi_k \cdot \theta_{ki} \le C,\ i = 1,2,\ldots,N_k,\ k = 1,2,3,\ldots. \tag{4.13}$$

We now turn to verification of (B1) - (B5). (B1) and (C1) are the same, and so are (B2) and (C2). (B3) is a consequence of the following two estimates, which hold when θ is normalized;

$$\frac{\xi_k}{n_k} \cdot \sum_{k=1}^{N_k} f_{ki}(\xi_k) = \frac{\xi_k}{n_k} \cdot \sum_{k=1}^{N_k} \theta_{ki} \cdot e^{-\theta_{ki} \cdot \xi_k} \ge (\text{by } 4.13) \ge \frac{\xi_k \cdot e^{-C}}{n_k} \cdot \sum_{k=1}^{N_k} \theta_{ki} \ge (\text{by } 4.12) \ge e^{-C}, \tag{4.14}$$

$$\frac{\xi_k}{n_k} \cdot \sum_{k=1}^{N_k} \theta_{ki} \cdot e^{-\theta_{ki} \cdot \xi_k} \le \frac{\xi_k}{n_k} \cdot \sum_{k=1}^{N_k} \theta_{ki} \le (\text{by (4.12)}) \le [\rho \cdot \delta \cdot (1 - \gamma^{-1})]^{-1}. \tag{4.15}$$

To verify (B4) use the general inequality $\left| e^{-x} - e^{-y} \right| \le |x - y| \cdot e^{\max\{|x|,|y|\}},\ -\infty < x, y < \infty$;

$$\left| f_{ki}(t \cdot \xi_k) - f_{ki}(s \cdot \xi_k) \right| = \theta_{ki} \cdot \left| e^{-\theta_{ki} \cdot t \cdot \xi_k} - e^{-\theta_{ki} \cdot s \cdot \xi_k} \right| \le |t - s| \cdot e^{\theta_{ki} \cdot \xi_k \cdot \max\{|t|,|s|\}}. \tag{4.16}$$

From (4.16) and (4.13) it is seen that (B4) is satisfied. To verify (B5), finally, we use the inequality $1 - e^{-x} \ge x \cdot (1 - e^{-x_0})/x_0,\ 0 \le x \le x_0$, in combination with (4.13);

$$F_{ki}(\xi_k) \cdot (1 - F_{ki}(\xi_k)) = (1 - e^{-\theta_{ki} \cdot \xi_k}) \cdot e^{-\theta_{ki} \cdot \xi_k} \ge \theta_{ki} \cdot \xi_k \cdot \{(1 - e^{-C})/C\} \cdot e^{-\theta_{ki} \cdot \xi_k} = C' \cdot \xi_k \cdot f_{ki}(\xi_k).$$

Thereby the theorem is proved. ♦

5 A General Limit Result for Passage Variables

5.1 Some notation and terminology

We start by settling some notation and terminology to be used in this section. Let $[t_0, t_1]$, $t_0 < t_1$, be a specified interval, and $z(t)$ a function on $[t_0, t_1]$. $M(z)$ denotes the *supremum* of $|z(t)|$ on $[t_0, t_1]$. The *modulus of variation* for z on $[t_0, t_1]$, $w(\Delta; z)$, is;

$$w(\Delta; z) = \sup\{|z(s) - z(t)| : |s - t| \le \Delta,\ t_0 \le s, t \le t_1\}. \tag{5.1}$$

A sequence $\{z_k(t)\}_{k=1}^{\infty}$ of functions on $[t_0, t_1]$ is *equicontinuous* if there is a function $w(\Delta)$, $\Delta > 0$, with $w(\Delta) \to 0$ as $\Delta \to 0$, such that $w(\Delta; z_k) \le w(\Delta)$, $0 < \Delta \le t_1 - t_0$, $k = 1,2,3,\ldots$. A sequence $\{Q_k\}_{k=1}^{\infty}$ of random variables is *bounded in probability*, if $\sup_k P(|Q_k| \ge M) \to 0$, as $M \to \infty$. A sequence $\{Z_k\}_{k=1}^{\infty}$ of stochastic process on $[t_0, t_1]$ satisfies *condition (T)* if;

(T) For any sequence $\{\Delta_k\}_{k=1}^{\infty}$ with $\Delta_k \to 0$, $w(\Delta_k; Z_k) \xrightarrow{P} 0$, as $k \to \infty$. (5.2)

Condition (T) is closely related to tightness properties of $\{\text{Law}(Z_k)\}_{k=1}^{\infty}$, a topic which is treated in depth in Billingsley (1968). Below we state a sufficient criterion for condition (T). Even if it differs in formulation from Theorem 12.3 in Billingsley (1968), the proof of that theorem contains all vital ingredients for verification of the criterion.

Criterion for Condition (T): Sufficient for a sequence $Z_1, Z_2, Z_3, \dots$ of separable stochastic processes to satisfy condition (T) is that for some $C<\infty$, and for each fixed $\Delta>0$;

$$\overline{\lim_{k\to\infty}} E\big(Z_k(t+\Delta)-Z_k(t)\big)^4 \le C\cdot\Delta^2, \ \ t_0\le t, t+\Delta\le t_1. \tag{5.3}$$

5.2 A limit theorem for passage variables

An interval $[t_0, t_1]$ is specified. For $k = 1, 2, 3, \dots$, V_k and U_k are stochastic processes on $[t_0, t_1]$, y_k and x_k are ordinary functions on $[t_0, t_1]$, and ε_k is a positive real number. The stochastic processes Y_k and X_k are defined by;

$$Y_k(t) = y_k(t) + \varepsilon_k \cdot V_k(t), \ \ t_0 \le t \le t_1, \tag{5.4}$$

$$X_k(t) = x_k(t) + \varepsilon_k \cdot U_k(t), \ \ t_0 \le t \le t_1. \tag{5.5}$$

For a real τ_k, the *level* τ_k *passage times* for y_k and Y_k, denoted t_k^* and T_k^* (often referred to as *inverses*) are defined by;

$$t_k^* = \inf\{t\colon y_k(t)\ge\tau_k\}, \ \ T_k^* = \inf\{t\colon Y_k(t)\ge\tau_k\}, \ \text{with the convention } \inf\emptyset = t_1. \tag{5.6}$$

Our interest concerns the asymptotic behavior (as $k\to\infty$) of the random variable

$$X_k(T_k^*) = \text{the } \textit{value of } X_k \textit{ when } Y_k \textit{ passes through the level } \tau_k. \tag{5.7}$$

THEOREM 5.1: For $k=1,2,3,\dots$, let y_k, x_k, ε_k, Y_k, X_k, τ_k, t_k^* and T_k^* be as specified above. Provided that conditions (A1) - (A8) below are fulfilled, we have;

$$\frac{X_k(T_k^*) - x_k(t_k^*)}{\varepsilon_k} - \left(U_k(t_k^*) - \frac{x_k'(t_k^*)}{y_k'(t_k^*)}\cdot V(t_k^*)\right) \xrightarrow{P} 0, \ \text{as } k\to\infty. \tag{5.8}$$

(A1) $\varepsilon_k \to 0$ as $k\to\infty$.

(A2) $\underline{\lim}_{k\to\infty} \min(\tau_k - y_k(t_0), y_k(t_1) - \tau_k) > 0$.

(A3) y_k has derivative y_k' on $[t_0, t_1]$, and $\underline{\lim}_{k\to\infty} \inf_{t_0\le t\le t_1} y_k'(t) > 0$.

(A4) $\{y_k'\}_{k=1}^{\infty}$ is equicontinuous on $[t_0, t_1]$.

(A5) x_k has derivative x'_k on $[t_0, t_1]$, and $\{x_k'\}_{k=1}^{\infty}$ is uniformly bounded and equicontinuous on $[t_0, t_1]$.

(A6) For some $t_k \in [t_0, t_1]$, $k=1,2,3,\dots$, $\{V_k(t_k)\}_{k=1}^{\infty}$ is bounded in probability.

(A7) $\{V_k\}_{k=1}^{\infty}$ satisfies condition (T).

(A8) $\{U_k\}_{k=1}^{\infty}$ satisfies condition (T).

Remark 5.1: When (A2) - (A4) are met, $y_k(t)$ is continuous and strictly increasing on $[t_0, t_1]$, and it crosses the level τ_k on $[t_0, t_1]$. Then t_k^* can be defined in the more transparent way as the *unique* solution to the equation (in t) $y_k(t) = \tau_k$. In particular; $y_k(t_k^*) = \tau_k$. ♦

Remark 5.2: If $t_k^* = t^*$ for $k=1,2,3,\dots$, where t^* is an interior point in $[t_0, t_1]$, then condition (A3) implies that (A2) is satisfied. ♦

Remark 5.3: A main aspect on the result (5.8) is that it tells that the two terms have the same asymptotic behavior. As a consequence, a limit result for one of the terms holds also for the other. Usually, the term within brackets is simplest to study, and it provides a mean for studying the other term. This use of (5.8) is illustrated in Section 6. ♦

The crucial part of the proof of Theorem 5.1 is stated in the following Lemma 5.1, where we continue to use notation and assumptions as in Theorem 5.1. The situation in Lemma 5.1 is, however, more special. It concerns just one k-situation, and we omit the index k. Moreover, also V and U are presumed to be ordinary (= non-random) functions. Hence, in Lemma 5.1 Y, X, y, x, V and U are real-valued, bounded functions on an interval $[t_0, t_1]$, $t_0 < t_1$, which are related in accordance with (5.4) and (5.5), ε and τ being real numbers, $\varepsilon > 0$. The level τ passage times for y and Y, t^* and T^*, are defined in accordance with (5.6). Furthermore, we assume that the following counterparts to (A2), (A3) and (A5) are in force;

$$y(t_0) < \tau < y(t_1), \tag{5.9}$$

The functions y and x have continuous derivatives y' and x' on $[t_0, t_1]$. (5.10)

For some $m > 0$, $y'(t) \geq m > 0$, $t_0 \leq t \leq t_1$. (5.11)

The proof of the following Lemma 5.1 is deferred until the next sub-section.

LEMMA 5.1: Let notation be as stated above. Assume that (5.9) - (5.11) are in force. Define c by (5.12), and presume that $\varepsilon > 0$ is small enough for (5.13) and (5.14) to hold;

$$c = 2 \cdot M(V)/m. \tag{5.12}$$

$$w(\varepsilon \cdot c; y') \leq m/3, \tag{5.13}$$

$$\varepsilon \leq \min(\tau - y(t_0), y(t_1) - \tau)/2 \cdot M(V). \tag{5.14}$$

For $R(\varepsilon)$ as defined implicitly below, (5.16) then holds;

$$X(T^*) = x(t^*) + \varepsilon \cdot \left(U(t^*) - \frac{x'(t^*)}{y'(t^*)} \cdot V(t^*) \right) + \varepsilon \cdot R(\varepsilon), \tag{5.15}$$

$$|R(\varepsilon)| \leq \frac{c \cdot M(x')}{m} \cdot w(\varepsilon \cdot c; y') + c \cdot w(\varepsilon \cdot c; x') + \frac{2 \cdot M(x')}{m} \cdot w(\varepsilon \cdot c; V) + w(\varepsilon \cdot c; U). \tag{5.16}$$

Remark 5.4: If we add to the other conditions in Lemma 5.1 that also U and V are continuous on $[t_0, t_1]$, the bound in (5.16) tells that $R(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence, the remainder term $\varepsilon \cdot R(\varepsilon)$ in (5.15) becomes negligible as ε tends to 0. ♦

Proof of Theorem 5.1: Set, in accordance with (5.15);

$$R_k(\varepsilon_k) = \frac{X_k(T_k^*) - x_k(t_k^*)}{\varepsilon_k} - \left(U_k(t_k^*) - \frac{x'_k(t_k^*)}{y'_k(t_k^*)} \cdot V_k(t_k^*) \right), \quad k = 1, 2, 3, \dots . \tag{5.17}$$

(A2), (A3) and (A5) imply that there are $\rho > 0$, $m > 0$ $M < \infty$, and $k_0 < \infty$, such that;

$$y_k(t_0) + \rho < \tau_k < y_k(t_1) - \rho, \quad y'_k(t) \geq m > 0, \ t_0 \leq t \leq t_1, \quad M(x'_k) \leq M, \quad k \geq k_0. \tag{5.18}$$

Fix a $\delta > 0$. From (A6) and (A7) we conclude that there is a $B_\delta < \infty$, and a $k_1(\delta) < \infty$, such that;

$$P(|M(V_k)| \leq B_\delta) \geq 1 - \delta, \quad k \geq k_1(\delta). \tag{5.19}$$

Set, in accordance with (5.12);

$$c_\delta = 2 \cdot B_\delta / m. \tag{5.20}$$

By virtue of (A1) and (A4) we have for some $k_2(\delta) < \infty$;

$$w(\varepsilon_k \cdot c_\delta; y'_k) \leq m/3, \quad k \geq k_1(\delta). \tag{5.21}$$

By (A1) and the left inequality in (5.18), the following bound holds on the set $\{|M(V_k)|) \leq B_\delta\}$ for some $k_3(\delta) < \infty$;

$$\varepsilon_k \leq \min(\tau_k - y_k(t_0), y_k(t_1) - \tau_k)/2 \cdot B_\delta, \quad k \geq k_3(\delta). \tag{5.22}$$

From (5.21) and (5.22) it is seen that Lemma 5.1 applies on the set $\{|M(V_k)| \leq B_\delta\}$ for k sufficiently large. Hence, by (5.16);

$$|R_k(\varepsilon_k)| \leq \frac{c_\delta \cdot M}{m} \cdot w(\varepsilon_k \cdot c_\delta; y'_k) + c_\delta \cdot w(\varepsilon_k \cdot c_\delta; x') + \frac{2 \cdot M}{m} \cdot w(\varepsilon_k \cdot c_\delta; V_k) + w(\varepsilon_k \cdot c_\delta; U_k),$$
$$\text{holds on } \{|M(V_k)| \leq B_\delta\}, \text{ for } k \geq \max\{k_0, k_1(\delta), k_2(\delta), k_3(\delta)\}. \tag{5.23}$$

By (A1), (A4) and (A5), the two first terms in (5.23) tend to 0 as $k \to \infty$. By (A7), (A8) and the definition of condition (T), the two last terms vanish on the set $\{|M(V_k)| \leq B_\delta\}$ as $k \to \infty$. By (5.19), $\{|M(V_k)| \leq B_\delta\}$ has probability $\geq 1 - \delta$. Hence, for every fixed $\gamma > 0$, $\overline{\lim}_{k \to \infty} P(|R_k(\varepsilon_k)| \geq \gamma) \leq \delta$. Since δ is arbitrary, the claim in the theorem follows. ♦

5.3 Proof of Lemma 5.1

First we list a well-known result concerning *Taylor approximation.* (5.24) introduces a notation for the remainder term in a one-step Taylor expansion. (5.25) presents an estimate of it, which follows from the mean value theorem. As before, w denotes modulus of variation.

$$\text{With } \rho(t,s;z) \text{ defined by; } z(t) = z(s) + (t - s) \cdot z'(s) + \rho(t,s;z), \tag{5.24}$$

$$|\rho(t,s;z)| \leq |t - s| \cdot w(|t-s|; z'). \tag{5.25}$$

Next we prepare the proof of Lemma 5.1 with two auxiliary lemmas.

LEMMA 5.2 Let notation and assumptions be as in Lemma 5.1. Then;

$$|T^* - t^*| \leq \varepsilon \cdot c. \tag{5.26}$$

Proof: By (5.10), (5.11) and (5.9), y is continuous, strictly increasing, and crosses the level τ on the interval $[t_0, t_1]$. Hence, t^* is the unique solution to the equation $y(t) = \tau$. In particular;

$$y(t^*) = \tau. \tag{5.27}$$

By (5.4), (5.10) and (5.24) we have, provided that $t_0 \leq t^*, t^* + h \leq t_1$;

$$Y(t^*+h) = y(t^*+h) + \varepsilon \cdot V(t^*+h) = \{\text{by Taylor expansion of } y(t^*+h)\} =$$
$$= y(t^*) + h \cdot y'(t^*) + \rho(t^*+h, t^*; y) + \varepsilon \cdot V(t^*+h). \tag{5.28}$$

From (5.28), (5.27) and (5.25) we get for $h > 0$; $Y(t^*+h) \geq \tau + h \cdot m - h \cdot w(h; y') - \varepsilon \cdot |V(t^*+h)|$. Now, provided that $0 < h \leq \varepsilon \cdot c$, this estimate can, by (5.13), be continued to;

$$Y(t^*+h) \geq \tau + h \cdot m - h \cdot m/3 - \varepsilon \cdot M(V) > \tau + h \cdot m/2 - \varepsilon \cdot M(V). \tag{5.29}$$

By setting $h = \varepsilon \cdot c$ in (5.29) and recalling (5.12), it is seen that $Y(t^*+\varepsilon \cdot c) > \tau$, provided that $t^*+\varepsilon \cdot c \leq t_1$. It may happen, though, that $t^*+\varepsilon \cdot c > t_1$. If so, note that $Y(t_1) = y(t_1) + \varepsilon \cdot V(t_1)$ and (5.14) imply that $Y(t_1) > \tau$. All in all this implies that $Y(\min(t^* + \varepsilon \cdot c, t_1)) > \tau$.

By very similar reasoning, it can be shown that $Y(\max(t^* - \varepsilon \cdot c, t_0)) < \tau$. Hence, Y has a level τ passage on the interval $[\max(t^* - \varepsilon \cdot c, t_0), \min(t^* + \varepsilon \cdot c, t_1)]$, which is included in $t^* \pm \varepsilon \cdot c$. Thereby (5.26) is proved. ♦

LEMMA 5.3: Let notation and assumptions be as in Theorem 5.1, and define $Q(\varepsilon)$ by (5.30). Then (5.31) holds.

$$T^* = t^* - \frac{\varepsilon}{y'(t^*)} \cdot V(t^*) + \varepsilon \cdot Q(\varepsilon). \tag{5.30}$$

$$|Q(\varepsilon)| \leq [c \cdot w(\varepsilon \cdot c; y') + 2 \cdot w(\varepsilon \cdot c; V)]/m. \tag{5.31}$$

Proof: Since Y need not be continuous nor increasing on $[t_0, t_1]$, $Y(T^*)$ and τ are not as simply related as $y(t^*)$ and τ. An analog to (5.27) is stated in (5.32), where $w(0+; z)$ denotes the limit of $w(\Delta; z)$ as $\Delta \to 0$, which also can be interpreted as the maximal jump for z on $[t_0, t_1]$.

$$|Y(T^*) - \tau| \leq w(0+; Y) = \{\text{by (5.4) and the fact that y is continuous}\} = \varepsilon \cdot w(0+; V). \tag{5.32}$$

By (5.4) and (5.24) we have; $Y(T^*) = y(T^*) + \varepsilon \cdot V(T^*) = \{\text{by Taylor expansion of } y(T^*)\} =$

$$= y(t^*) + (T^* - t^*) \cdot y'(t^*) + \rho(T^*, t^*; y) + \varepsilon \cdot V(t^*) + \varepsilon \cdot (V(T^*) - V(t^*)). \tag{5.33}$$

Rearrangement in (5.33), (5.27) and comparison with (5.30) yields;

$$\varepsilon \cdot Q(\varepsilon) = [(Y(T^*) - \tau) - \rho(T^*, t^*; y) - \varepsilon \cdot (V(T^*) - V(t^*))]/y'(t^*). \tag{5.34}$$

By applying (5.32), (5.25) and (5.26) in (5.34), we get;

$$|Q(\varepsilon)| \leq [\varepsilon \cdot w(0+; V) + |T^* - t^*| \cdot w(|T^* - t^*|; y') + \varepsilon \cdot w(|T^* - t^*|; V)]/(\varepsilon \cdot m). \tag{5.35}$$

By using (5.26) and $w(0+; z) \leq w(\Delta; z)$ in (5.35), the bound (5.31) is obtained. ♦

Proof of Lemma 5.1: By (5.5) and (5.24); $X(T^*) = x(T^*) + \varepsilon \cdot U(T^*) = \{\text{Taylor expansion}$

$$\text{of } x(T^*)\} = x(t^*) + (T^* - t^*) \cdot x'(t^*) + \rho(T^*, t^*; x) + \varepsilon \cdot U(t^*) + \varepsilon \cdot (U(T^*) - U(t^*)). \tag{5.36}$$

Insertion of $T^* - t^*$ according to (5.30) into (5.36) and comparison with (5.15) yields;

$$R(\varepsilon) = x'(t^*) \cdot Q(\varepsilon) + \rho(T^*, t^*; x)/\varepsilon + (U(T^*) - U(t^*)). \tag{5.37}$$

By employing (5.25) in (5.37) we get

$$|R(\varepsilon)| = |x'(t^*)| \cdot |Q(\varepsilon)| + |T^* - t^*| \cdot w(|T^* - t^*|; x')/\varepsilon + w(|T^* - t^*|; U). \tag{5.38}$$

Now, application of (5.31) and (5.26) in (5.38) leads to (5.16). ♦

6 Proof of Theorem 3.1

6.1 The proof

The instrumental tool in the proof will be Theorem 5.1, and we start with some preparations for its application. For the ordering variables $Q_1, Q_2, ..., Q_N$ in Definition 2.1, introduce the following stochastic processes $H_i(s)$ on $0 \leq s < \infty$, where H_i "counts when Q_i occurs". ($\mathbf{1}_A$ denotes the indicator of the set A);

$$H_i(s) = \mathbf{1}_{\{Q_i \leq s\}}, \quad 0 \leq s < \infty, \quad i = 1, 2, ..., N. \tag{6.1}$$

$H_i(s)$ is a 0-1 random variable with expected value $F_i(s)$, which yields;

$$E[H_i(s)] = F_i(s) \quad \text{and} \quad Var[H_i(s)] = F_i(s) \cdot (1 - F_i(s)), \quad 0 \leq s < \infty. \tag{6.2}$$

Define the stochastic processes $J(t)$ and $L(t; \mathbf{z})$ as follows, where $\mathbf{z} = (z_1, z_2 ..., z_N)$ are given real numbers and $\xi > 0$ is arbitrary (but fixed);

$$J(t) = \sum_{i=1}^{N} H_i(t \cdot \xi) \quad \text{and} \quad L(t; \mathbf{z}) = \sum_{i=1}^{N} z_i \cdot H_i(t \cdot \xi), \quad 0 \leq t < \infty. \tag{6.3}$$

Let T^* be the time when $J(t)$ hits the level n, and $A(n; \mathbf{z})$ the value of $L(t; \mathbf{z})$ at the hitting time;

$$T^* = \inf\{t: J(t) = n\} \quad \text{and} \quad A(n;\mathbf{z}) = L(T^*;\mathbf{z}). \tag{6.4}$$

The following relation (6.5) should be evident upon some thought. It provides the link between order sampling and passage variables, which allows application of Theorem 5.1.

The sample sum $S(n;\mathbf{z})$ under $OS(n;\mathbf{F})$ has the same distribution as $A(n;\mathbf{z})$. (6.5)

We adhere to the sequence version in Theorem 3.1. For $k = 1,2,3,..$, let $J_k(t)$, $L_k(t;\mathbf{z}_k)$, T_k^* and $A_k(n_k;\mathbf{z}_k)$ be in accordance with what is said above, and let ξ_k, μ_k, ϕ_k and σ_k be in accordance with (3.2) - (3.5). By virtue of (6.5) we have;

$$\text{Law}[(S_k(n_k;\mathbf{z}_k) - \mu_k)/\sigma_k] \text{ agrees with } \text{Law}[(A_k(n_k;\mathbf{z}_k) - \mu_k)/\sigma_k],\ k = 1,2,3,\ldots . \tag{6.6}$$

As will be proved, Theorem 5.1 yields that $(A_k(n_k;\mathbf{z}_k) - \mu_k)/\sigma_k$ is asymptotically N(0,1) - distributed under the conditions in Theorem 3.1. Having shown that, (6.6) tells that the same holds for $(S_k(n_k;\mathbf{z}_k) - \mu_k)/\sigma_k$, and Theorem 3.1 is thereby proved.

When employing Theorem 5.1, the entities $Y_k, X_k, x_k, y_k, V_k, U_k, \varepsilon_k$ and τ_k are chosen as stated below. $[t_0, t_1] = [1-\delta, 1+\delta]$, where δ is that in (B4). When checking (6.7) - (6.10), recall the left equality in (6.2), and that ^{c} = centering at expectation.

$$Y_k(t) = \frac{1}{n_k}\sum_{i=1}^{N} H_{ki}(t\cdot\xi_k) = y_k(t) + \frac{1}{\sqrt{n_k}}\cdot V_k(t),\ t_0 \le t \le t_1, \tag{6.7}$$

$$y_k(t) = \frac{1}{n_k}\sum_{i=1}^{N_k} F_{ki}(t\cdot\xi_k),\quad V_k(t) = \frac{1}{\sqrt{n_k}}\sum_{i=1}^{N_k} H_{ki}(t\cdot\xi_k)^c,\ t_0 \le t \le t_1, \tag{6.8}$$

$$X_k(t) = \frac{1}{\sqrt{n_k}}\sum_{i=1}^{N_k}\frac{z_{ki}-\phi_k}{\sigma_k}\cdot H_{ki}(t\cdot\xi_k) = x_k(t) + \frac{1}{\sqrt{n_k}}\cdot U_k(t),\ t_0 \le t \le t_1, \tag{6.9}$$

$$x_k(t) = \frac{1}{\sqrt{n_k}}\sum_{i=1}^{N_k}\frac{z_{ki}-\phi_k}{\sigma_k}\cdot F_{ki}(t\cdot\xi_k),\quad U_k(t) = \sum_{i=1}^{N_k}\frac{z_{ki}-\phi_k}{\sigma_k}\cdot H_{ki}(t\cdot\xi_k)^c,\ t_0 \le t \le t_1, \tag{6.10}$$

$$\varepsilon_k = 1/\sqrt{n_k},\ k=1,2,3,\ldots , \tag{6.11}$$

$$\tau_k = 1,\quad k = 1,2,3,\ldots . \tag{6.12}$$

From (6.12) and (3.6) follows that t_k^* in (5.6) here takes the value $t_k^* = 1$.

The proof of Theorem 3.1 comprises two main parts, which are stated as separate lemmas below. Together the two lemmas yield that $(A_k(n_k;\mathbf{z}_k) - \mu_k)/\sigma_k$ is asymptotically N(0,1) - distributed under (B1) - (B5), which is the claim in Theorem 3.1. In addition, Section 6.2 contains two lemmas with auxiliary results, which will be used in the sequel.

LEMMA 6.1: Under conditions (B1) - (B5) in Theorem 3.1, we have;

$$\frac{A_k(n_k;\mathbf{z}_k) - \mu_k}{\sigma_k} - U_k(1) \xrightarrow{P} 0,\ \text{as } k \to \infty. \tag{6.13}$$

LEMMA 6.2: Under condition (B2) in Theorem 3.1, we have;

$$\text{Law}[U_k(1)] \Rightarrow N(0,1),\ \text{as } k \to \infty. \tag{6.14}$$

We start by proving Lemma 6.2, which is simplest.

Proof of Lemma 6.2: Since $Q_{k1}, Q_{k2}, \ldots, Q_{kN}$ are independent, so are the processes $H_{ki}(t)$, $i = 1, 2, \ldots, N_k$. Hence, cf. (6.10),

$$U_k(1) = \sum_{i=1}^{N_k}\frac{z_{ki}-\phi_k}{\sigma_k}\cdot H_{ki}(\xi_k)^c \tag{6.15}$$

is a sum of independent random variables with means 0. By using (6.2) and (3.5) it is readily checked that the variance of $U_k(1)$ is 1. To verify (6.14) we apply Lyapunov's condition, in 4:th moments version, for the central limit theorem for independent variables.

$$\sum_{i=1}^{N_k} E\left(\frac{z_{ki}-\phi_k}{\sigma_k}\cdot H_{ki}(\xi_k)^c\right)^4 \le \text{(by Lemma 6.3)} \le \frac{1}{4}\cdot\sum_{i=1}^{N_k}\frac{(z_{ki}-\phi_k)^4}{\sigma_k^4}\cdot F_{ki}(\xi_k)\cdot(1-F_{ki}(\xi_k)) \le$$

$$\le \max_i \frac{(z_{ki}-\phi_k)^2}{\sigma_k^2}\cdot\sum_{i=1}^{N_k}\frac{(z_{ki}-\phi_k)^2}{\sigma_k^2}\cdot F_{ki}(\xi_k)\cdot(1-F_{ki}(\xi_k)) = \text{(by (3.5)=} \left(\max_i \frac{|z_{ki}-\phi_k|}{\sigma_k}\right)^2. \quad (6.16)$$

By (B2) the last term in (6.16) tends to 0 as $k\to\infty$, and Lemma 6.2 is proved. ♦

Proof of Lemma 6.1: First we derive expressions for the two terms in (5.8). By (6.3) and (6.4), it is readily checked that;

$$X_k(T_k^*) = \frac{L_k(T_k^*;z_k)-\phi_k\cdot J_k(T_k^*)}{\sigma_k\cdot\sqrt{n_k}} = \frac{A_k(n_k;z)-\phi_k\cdot n_k}{\sigma_k\cdot\sqrt{n_k}}. \quad (6.17)$$

Furthermore, (6.10), (3.3) and (3.4) yield;

$$x_k(t_k^*) = x_k(1) = \frac{\mu_k-\phi_k\cdot n_k}{\sigma_k\cdot\sqrt{n_k}}. \quad (6.18)$$

From (6.17) and (6.18) we get;

$$\frac{X_k(T_k^*)-x_k(t_k^*)}{\varepsilon_k} = \frac{A_k(n_k;z)-\mu_k}{\sigma_k}. \quad (6.19)$$

The formulas for y_k and x_k in (6.8) and (6.10), and the assumption that the F_{ki} have densities imply that y_k and x_k are differentiable on $[t_0, t_1]$, and;

$$y_k'(t) = \frac{\xi_k}{n_k}\cdot\sum_{i=1}^{N_k} f_{ki}(t\cdot\xi_k),\quad t_0\le t\le t_1, \quad (6.20)$$

$$x_k'(t) = \frac{\xi_k}{\sqrt{n_k}}\cdot\sum_{i=1}^{N_k}\frac{z_{ki}-\phi_k}{\sigma_k}\cdot f_{ki}(t\cdot\xi_k),\quad t_0\le t\le t_1. \quad (6.21)$$

From (6.21) and (3.4) follows that $x_k'(t_k^*) = x_k'(1) = 0$. Hence, the term to the right in (5.8) "collapses" to $U_k(1)$. This together with (6.19) yields that *if* Theorem 5.1 applies in the present situation, it leads to (6.13). It remains, though, to show that Theorem 5.1 in fact does apply, which we do by proving that conditions (B1)-(B5) imply (A1)-(A8).

(A1) follows from (6.11) and (B1). (A3) is the left part of (B3). By Remark 5.2, (A2) follows from (A3), $t_k^*=1$, $k=1,2,..$, and the choice of the interval $[t_0, t_1]$. To verify (A4) we use (6.20);

$$|y_k'(t)-y_k'(s)| \le \frac{\xi_k}{n_k}\cdot\sum_{i=1}^{N_k}|f_{ki}(t\cdot\xi_k)-f_{ki}(s\cdot\xi_k)| \le \text{by (B4)} \le C\cdot w(|t-s|)\cdot\frac{\xi_k}{n_k}\cdot\sum_{i=1}^{N_k} f_{ki}(\xi_k). \quad (6.22)$$

Formula (6.22), the right hand part of (B3) and the properties of w yield that (A4) is satisfied. When considering (A5), we first show that $\{x_k'(1)\}_{k=1}^{\infty}$ is bounded, and we start from (6.21).

$$|x_k'(1)| = \left|\frac{\xi_k}{\sqrt{n_k}}\cdot\sum_{i=1}^{N_k}\frac{z_{ki}-\phi_k}{\sigma_k}\cdot f_{ki}(\xi_k)\right| \le \text{(by Schwarz' inequality)} \le$$

$$\le\sqrt{\sum_{i=1}^{N_k}\left(\frac{z_{ki}-\phi_k}{\sigma_k}\right)^2\cdot F_{ki}(\xi_k)\cdot(1-F_{ki}(\xi_k))}\cdot\sqrt{\frac{\xi_k^2}{n_k}\cdot\sum_{i=1}^{N_k}\frac{f_{ki}(\xi_k)^2}{F_{ki}(\xi_k)\cdot(1-F_{ki}(\xi_k))}}. \quad (6.23)$$

By (3.5), the left square root in (6.23) equals 1. By using (B5) in the right hand square root, we arrive at the following estimate, which together with (B3) shows that $\{x_k'(1)\}_{k=1}^{\infty}$ is bounded;

$$|x_k'(1)| \le \sqrt{\frac{\xi_k}{\rho \cdot n_k} \cdot \sum_{i=1}^{N_k} f_{ki}(\xi_k)}\,. \tag{6.24}$$

Next we show that $\{x_k'(t)\}_{k=1}^{\infty}$ is equicontinuous on $[t_0, t_1]$, and we start from (6.21);

$$|x_k'(t) - x_k'(s)| = \left| \frac{\xi_k}{\sqrt{n_k}} \cdot \sum_{i=1}^{N_k} \frac{z_{ki} - \phi_k}{\sigma_k} \cdot (f_{ki}(t \cdot \xi_k) - f_{ki}(s \cdot \xi_k)) \right| \le \text{(by Schwarz' inequality and (3.5))} \le$$

$$\le \sqrt{\frac{\xi_k^2}{n_k} \sum_{i=1}^{N_k} \frac{(f_{ki}(t \cdot \xi_k) - f_{ki}(s \cdot \xi_k))^2}{F_{ki}(\xi_k) \cdot (1 - F_{ki}(\xi_k))}} \le \text{(by (B4) and (B5))} \le \sqrt{\frac{C^2 \cdot w(|t-s|)^2 \cdot \xi_k}{\rho \cdot n_k} \sum_{i=1}^{N_k} f_{ki}(\xi_k)}\,. \tag{6.25}$$

By (B3), (6.25) tells that $\{x_k'(t)\}_{k=1}^{\infty}$ is equicontinuous on $[t_0, t_1]$. This together with $\{x_k'(1)\}_{k=1}^{\infty}$ being bounded verifies (A5). The following estimate implies (A6) (recall (6.8), (6.2) and (3.6));

$$\mathrm{Var}[V_k(1)] = \frac{1}{n_k} \cdot \sum_{i=1}^{N_k} \mathrm{Var}[H_{ki}(\xi_k)] = \frac{1}{n_k} \sum_{i=1}^{N_k} F_{ki}(\xi_k) \cdot (1 - F_{ki}(\xi_k)) \le \frac{1}{n_k} \sum_{i=1}^{N_k} F_{ki}(\xi_k) = 1\,. \tag{6.26}$$

To show that (A8) is met, we employ condition (5.3), and we start from (6.10);

$$\overline{\lim_{k\to\infty}} E[U_k(t) - U_k(s)]^4 = \overline{\lim_{k\to\infty}}\ E[\sum_{i=1}^{N_k} \frac{z_{ki} - \phi_k}{\sigma_k} \cdot (H_{ki}(t \cdot \xi_k)^c - H_{ki}(s \cdot \xi_k)^c)]^4 \le \text{(by Lemma 6.4)} \le$$

$$\le 6 \cdot \overline{\lim_{k\to\infty}} \sum_{i=1}^{N_k} \left(\frac{z_{ki} - \phi_k}{\sigma_k} \right)^4 \cdot E[\{H_{ki}(t \cdot \xi_k) - H_{ki}(s \cdot \xi_k)\}^c]^4 +$$

$$+ 6 \cdot \overline{\lim_{k\to\infty}} \left(\sum_{i=1}^{N_k} \left(\frac{z_{ki} - \phi_k}{\sigma_k} \right)^2 \cdot \mathrm{Var}[H_{ki}(t \cdot \xi_k) - H_{ki}(s \cdot \xi_k)] \right)^2. \tag{6.27}$$

Here we make an interlude to derive some auxiliary results.

Under (B4) there is a $C' < \infty$, such that for $t \in [t_0, t_1]$;

$$f_{ki}(t \cdot \xi_k) \le C' \cdot f_{ki}(\xi_k), \quad i = 1,2,\ldots,N_k, \quad k = 1,2,3,\ldots\,. \tag{6.28}$$

Check:

$$f_{ki}(t \cdot \xi_k) \le f_{ki}(\xi_k) + |f_{ki}(t \cdot \xi_k) - f_{ki}(\xi_k)| \le \text{by (B4)} \le f_{ki}(\xi_k) \cdot (1 + C \cdot w(|t-1|) \le C' \cdot f_{ki}(\xi_k)\,.$$

Under (B4) and (B5) there is a $C'' < \infty$, such that for $t, s \in [t_0, t_1]$;

$$|F_{ki}(t \cdot \xi_k) - F_{ki}(s \cdot \xi_k)| \le C'' \cdot |t-s| \cdot F_{ki}(\xi_k) \cdot (1 - F_{ki}(\xi_k))\,. \tag{6.29}$$

Check: Recall that F_{ki} is differentiable with continuous derivative f_{ki}. By the mean value theorem; $|F_{ki}(t \cdot \xi_k) - F_{ki}(s \cdot \xi_k)| = |t-s| \cdot \xi_k \cdot f_{ki}(\kappa)$, where κ is intermediate to $t \cdot \xi_k$ and $s \cdot \xi_k$. By applying first (6.28) and then (3.9), (6.29) is obtained.

We now return to (6.27). Note that $H_{ki}(t \cdot \xi_k) - H_{ki}(s \cdot \xi_k)$ is a 0 - 1 random variable with mean value $F_{ki}(t \cdot \xi_k) - F_{ki}(s \cdot \xi_k)$. By Lemma 6.3 and (6.29), (6.27) can be continued as follows;

$$\le 6 \cdot C'' \cdot |t-s| \cdot \overline{\lim_{k\to\infty}} \left(\max_i \frac{z_{ki} - \phi_k}{\sigma_k} \right)^2 \cdot \sum_{i=1}^{N_k} \left(\frac{z_{ki} - \phi_k}{\sigma_k} \right)^2 \cdot F_{ki}(\xi_k) \cdot (1 - F_{ki}(\xi_k) +$$

$$+ 6 \cdot (C'')^2 \cdot (t-s)^2 \cdot \overline{\lim_{k\to\infty}} \left(\sum_{i=1}^{N_k} \left(\frac{z_{ki} - \phi_k}{\sigma_k} \right)^2 \cdot F_{ki}(\xi_k) \cdot (1 - F_{ki}(\xi_k) \right)^2. \tag{6.30}$$

By (3.5) the sums in (6.30) equal 1. From this and (B2) it is seen that the first term in (6.30) equals 0, and the second equals $6 \cdot C'' \cdot (t-s)^2$. Now condition (5.3) tells that (A8) is satisfied. Verification of (A7) is similar but simpler, and some details are omitted. We start from (6.8);

$$\overline{\lim_{k\to\infty}} E[V_k(t) - V_k(s)]^4 = \overline{\lim_{k\to\infty}} E[\frac{1}{\sqrt{n_k}} \cdot \sum_{i=1}^{N_k} \{H_{ki}(t \cdot \xi_k) - H_{ki}(s \cdot \xi_k)\}^c]^4 \leq \text{(by Lemma 6.4)} \leq$$

$$\leq 6 \cdot \overline{\lim_{k\to\infty}} \frac{1}{n_k^2} \cdot \sum_{i=1}^{N_k} E[\{H_{ki}(t \cdot \xi_k) - H_{ki}(s \cdot \xi_k)\}^c]^4 + 6 \cdot \overline{\lim_{k\to\infty}} \left(\frac{1}{n_k} \cdot \sum_{i=1}^{N_k} Var[H_{ki}(t \cdot \xi_k) - H_{ki}(s \cdot \xi_k)] \right)^2 \leq$$

$$\leq 6 \cdot C'' \cdot |t-s| \cdot \overline{\lim_{k\to\infty}} \frac{1}{n_k^2} \cdot \sum_{i=1}^{N_k} F_{ki}(\xi_k) + 6 \cdot (C'')^2 \cdot (t-s)^2 \cdot \overline{\lim_{k\to\infty}} \left(\frac{1}{n_k} \cdot \sum_{i=1}^{N_k} F_{ki}(\xi_k) \right)^2. \qquad (6.31)$$

Now the estimate (6.31) together with (3.6) yields that (5.3) is satisfied, and (A7) is verified. Hence, Lemma 6.1, and thereby also Theorem 3.1 are proved. ♦

6.2 Two auxiliary results

The following results are certainly known, but they are proved for the sake of completeness.

LEMMA 6.3: For a 0-1 random variable Z with $P(Z=1)=p$, we have;

$$E(Z^c)^4 = E(Z-p)^4 \leq p \cdot (1-p)/4. \qquad (6.32)$$

Proof: By binomial expansion of $(Z-p)^4$ and use of $EZ = EZ^2 = EZ^3 = EZ^4 = p$, it follows that $E(Z^c)^4 = p \cdot (1-p) \cdot (3p^2 - 3p + 1) \leq p \cdot (1-p)/4$. ♦

LEMMA 6.4: Let $Z_1, Z_2, ..., Z_N$ be independent random variables. Then;

$$E\left(\sum_{i=1}^{N} Z_i^c\right)^4 \leq 6 \cdot [\sum_{i=1}^{N} E(Z_i^c)^4 + (\sum_{i=1}^{N} Var(Z_i))^2]. \qquad (6.33)$$

Proof: By binomial expansion of $(\sum_1^N Z_i^c)^4 = (\sum_1^{N-1} Z_i^c + Z_N^c)^4$ and taking expectation we get; $E(\sum_1^N Z_i^c)^4 = E(\sum_1^{N-1} Z_i^c)^4 + 6 \cdot Var(Z_N) \cdot \sum_1^{N-1} Var(Z_i) + E(Z_N^c)^4$. By employing induction in N, this identity readily yields (6.33). ♦

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