

# On Testing for Symmetry in Business Cycles

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and  
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# On Testing for Symmetry in Business Cycles

by

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## Abstract

Business cycle forecasting has become an important part of short and medium term economic planning. Such forecasting, however, is often very intricate, as business cycles are not at all periodic, just recurrent. Furthermore, they often include irregular timing and varying amplitudes. When patterns and relationships are very irregular there are no simple reliable business cycle forecasting procedures. In practice there is, somewhere, a limit for business cycle predictability, and it is often worthwhile to examine empirically the various theoretical regularity assumptions. One important regularity issue concerns the business cycle symmetry assumption. The present paper empirically tests the hypothesis of symmetry around business cycle turning points in some economic time series. Two test procedures are applied. One is based on the analysis of transition probabilities between expansion and recession regimes. The second procedure tests symmetry versus asymmetry through skewness statistics.

**Key words:** Business cycle, asymmetry, Markov process, maximum likelihood, Monte Carlo simulation, skewness, industrial production, unemployment.

## 1. Introduction

Business cycles refer to certain fluctuations in aggregate economic activities of nations. The cycles consist of expansions followed by recessions, contractions and revivals, leading to new periods of expansions, etc. As such changes in the economy substantially affect the performance of business firms as well as the more general aspects of the nation's economy, business cycle forecasting has become an important part of short and medium term economic planning.

Such forecasting is often very intricate, as business cycles are not at all periodic, just recurrent. Furthermore, they often include irregular timing and varying amplitudes. As the existence of cycles in economic aggregates entails that steady-state conditions are satisfied only exceptionally, it is obviously very difficult to develop a simple theory that encompasses the basic features of business cycles. These aggravating circumstances affect above all quantitative and more formalized forecasting approaches (e.g. those based on econometric models), but of course also less theoretically demanding judgemental forecasting procedures. We have to recognize that all types of forecasting procedures are more or less extrapolative in nature. Even judgemental forecasts are based on existing trends and patterns. When the patterns and relationships become too irregular, there are no simple reliable business cycle forecasting procedures. In practice there is, somewhere, a limit for business cycle predictability.

It is worthwhile to examine empirically the various theoretical regularity assumptions, referring to concepts such as stability, symmetry and linearity. The essential meaning of the concept varies with the forecasting situation, and in particular with the economic system in question, how it is defined and accordingly modeled. When economic systems are modeled by sets of behavioral relations, the analysis is generally based on assumptions of structural stability, formalized by parameter and functional form constancy assumptions (for conceptual definitions, see e.g. Westlund and Zackrisson, 1986). Such structural assumptions are always just approximately fulfilled, and often empirical-

ly rejected. Consequently, the structural variability must be identified and characterized, the models respecified, and the estimation and forecasting procedures designed to consider postulated structural variability characteristics (see Hackl and Westlund, 1988).

When business cycle forecasting is based on univariate time series analysis, regularity refers to e.g. homoscedasticity and to symmetry around cycle turning points. Cyclical amplitudes are to some extent related to the efficiency of stabilizing policies. Heteroscedasticity, in the sense that cycle variability decreases, may be the result of successful stabilizing activities (see Sheffrin, 1987). Whether business cycles tend to have predictable periodicities, or if they essentially are random walks without predictable values, is considered in Zarnowitz (1987). His basic findings indicate that the age of a business cycle phase is of minor significance when predicting its end. The variability in length of both expansions and contractions is large enough to complicate turning point forecasts (according to Zarnowitz, peak forecasting is particularly difficult). On the other hand Zarnowitz states that regularity manifests itself in dynamics, i.e. leads and lags among specific indicators are often reported to be sufficiently stable over time. His conclusions are based on comprehensive descriptive approaches, but his conclusions are not in line with the results obtained through transfer function analysis (see Westlund and Öhlén, 1989) which indicates significant lead variability among leading indicators and business cycle reference series.

One important regularity issue concerns the business cycle symmetry assumption. The interest in asymmetries in business cycles has increased during the last few years. Although Keynes (1936) among others proposed the existence of such asymmetries, stating that "the substitution of a downward for an upward tendency often takes place suddenly and violently, whereas there is, as a rule, no sharp turning point when an upward is substituted for a downward tendency", there are few theoretical or empirical studies of such asymmetries. Hicks (1950) suggested that cycles are generally asymmetric. Asymmetries in business cycles have been studied within the framework of the nonlinear theory of business cycles (see e.g. Chang and Smith, 1971, Rose, 1967) and



also within the framework of the so-called catastrophe theory (see e.g. Sussman and Zahler, 1978 and Varian 1979). Empirical tests have been done by e.g Neftçi (1984), Falk (1986) and by De Long and Summers (1984). Other empirical evidence includes work done by Blanchard and Watson (1984), who published evidence that business cycles are driven by many independent shocks rather than by a few infrequent major shocks. But despite the interest in and awareness of the need of nonlinear models of asymmetric behavior, the empirical evidence of asymmetry is limited.

Asymmetry can, of course, be defined in several ways. Roughly the interest of asymmetry has focused on the difference between a regime of expansion in some key variable, e.g. GNP, and a regime of contraction or recession. Keynes suggested that recessions in employment would be more violent but last for a shorter time period than expansions. If there are such asymmetries in key variables of the economy, they have to be identified and modeled. As has been pointed out by Neftçi, the existence of asymmetry demands a new theory of behavior. We must also expect that predictions using symmetry assumptions would be inferior, especially in the region of turning points of the business cycle.

The purpose of the present paper is to test empirically the hypothesis of symmetry around business cycle turning points in some economic time series. Two test procedures are applied. One is based on the analysis of transition probabilities between expansion and recession regimes. Chapter 2 briefly presents the technique and summarizes the basic results obtained by that procedure. The second procedure is straightforward and tests symmetry versus asymmetry through regular skewness statistics. The test and corresponding empirical results are found in Chapter 3. Chapter 4, finally, gives some conclusions.

## 2. Unequal transition probabilities and business cycle asymmetry

### 2.1 Introductory Comments

One way to analyze business cycles with respect to possible asymmetric behavior is to apply the test procedure suggested by Neftçi (1984). This analysis requires that the concept of 'asymmetry' is defined as follows:

"the business cycle reference time series is asymmetric if sequences of positive (or negative) differences (in general after seasonal and/or trend adjustments) are more likely than sequences of negative (or positive) differences."

Adjustments with respect to seasonal and trend components are basically made because the analysis is to focus on cycle properties. Furthermore, the test requires stationary data. Trend identification and trend elimination are far from trivial. Separating trend and cycle components is always more or less judgemental, and trend elimination may also remove part of the asymmetry dimension of the business cycle. Thus, trend elimination is expected to reduce the power of the test procedure. As the business cycle reference series Neftçi therefore selected various unemployment rates, as these series are more or less trend free (unemployment in the US was found to be asymmetric). Industrial production, the main indicator used in the present paper, involves positive trends, as do other production, investment and productivity indicators. Falk (1986) follows Neftçi, but besides unemployment rates he also uses non-stationary data, such as GNP, industrial production and investment in the US and in some other OECD countries. He therefore applies various trend elimination procedures, but does not observe any noticeable sensitivity in the test results when altering trend elimination approach. Falk's empirical findings do not suggest the existence of asymmetries, except for the case of US unemployment.

As monthly data probably will impose too many white noise errors to the data, thus reducing power, and annual data involve too few observations, also reducing power, the present study is based on quarterly data. It focuses on the quarterly Swedish industrial production index ( $IP_{80:I} = 1$ ), deseasonalized and detrended (with linear trend functions). The sample covers 1960:I – 1988:I. For comparative purposes, industrial production series for a number of other countries are also analyzed with respect to business cycle asymmetry. Furthermore, 'unemployment' (number of unemployed, UNP) is also analyzed. A conflict on the Swedish labor market during the second quarter of 1980 caused a missing value of the unemployment variable. We extrapolated the value by setting  $UNP_{80:II} = (UNP_{82:II} + UNP_{81:II})/2$ . The conflict also produced an outlier in the industrial production series (which decreased considerably during the conflict). The unemployment series covers the same period as industrial production. The original and detrended series are shown in Figures 1–4 below.

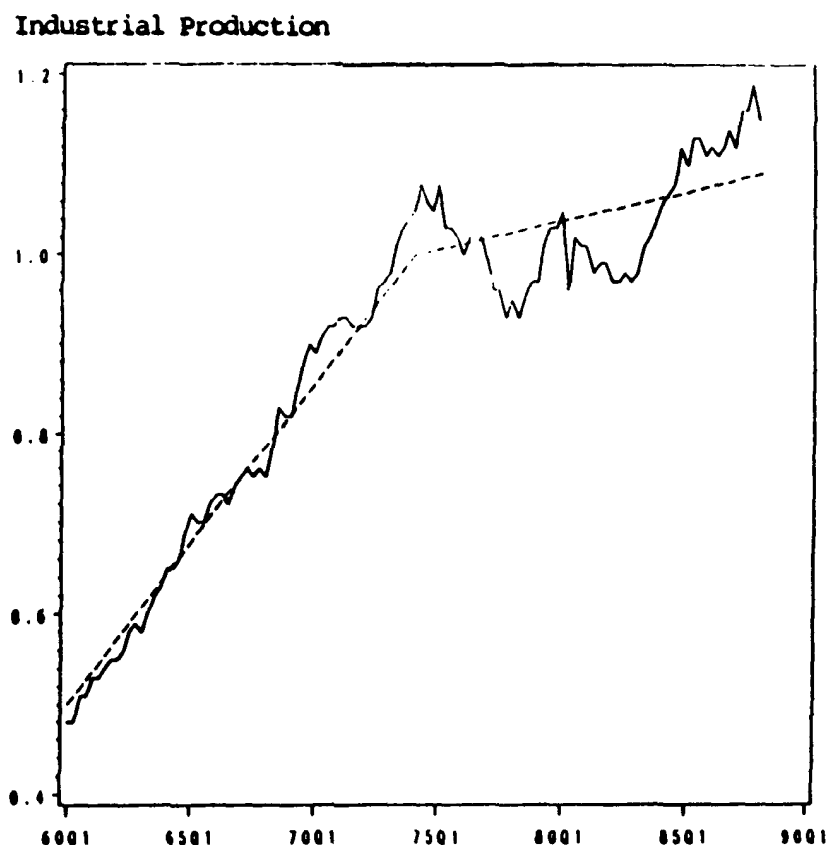


Fig. 1. Swedish Industrial Production and Linear Trends.  
Seasonally Adjusted Data. 1980:I=1.

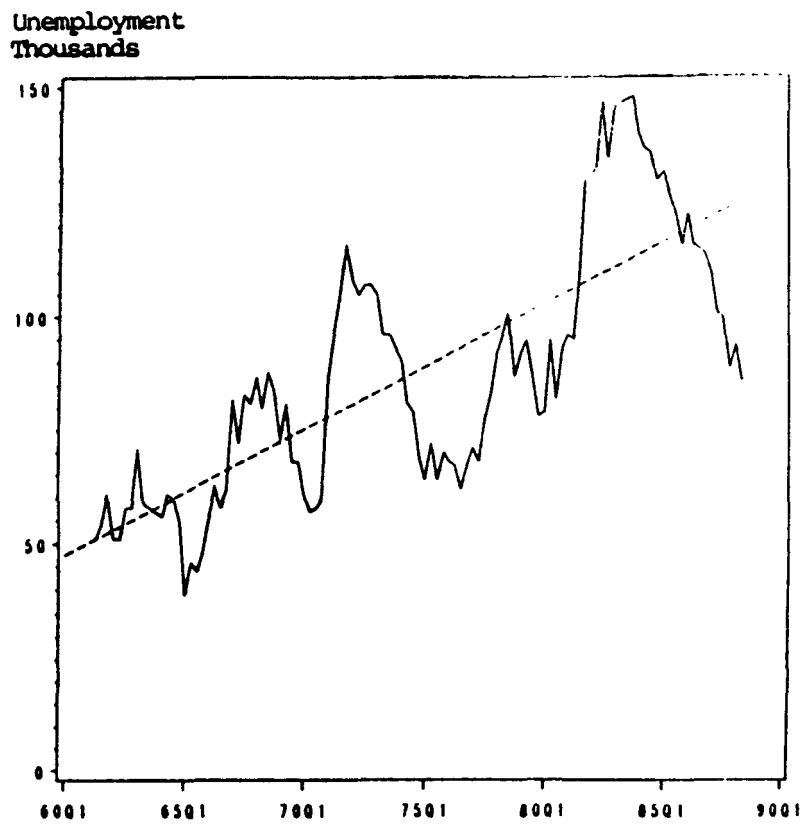


Fig. 2. Swedish Unemployment and Linear Trend. Number of Unemployed.

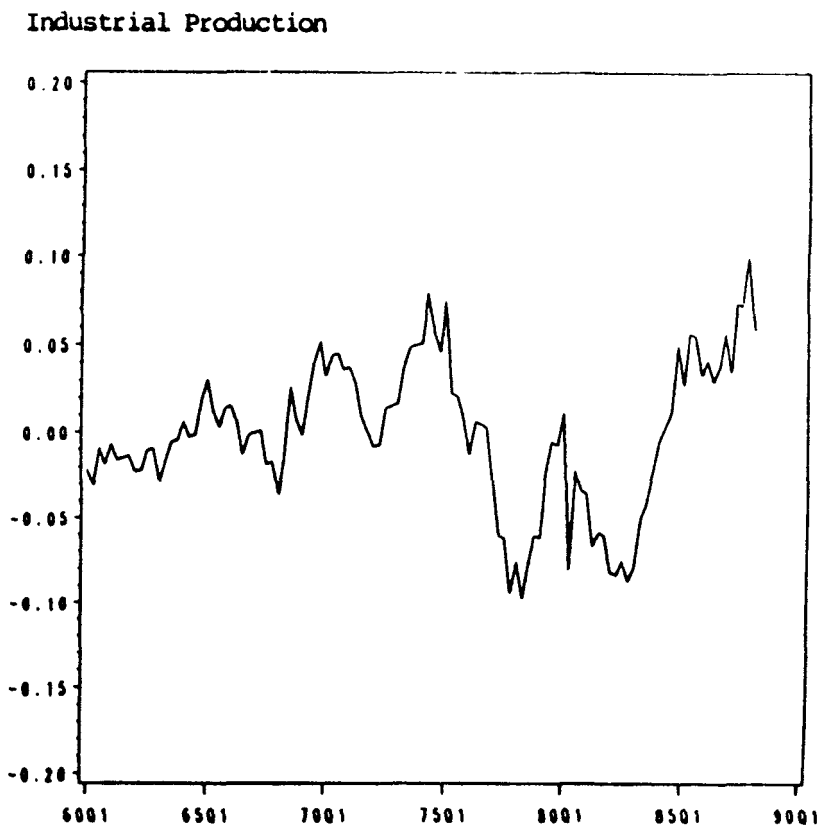


Fig. 3. Swedish Detrended Industrial Production Deviation from Trend.

Unemployment  
Thousands

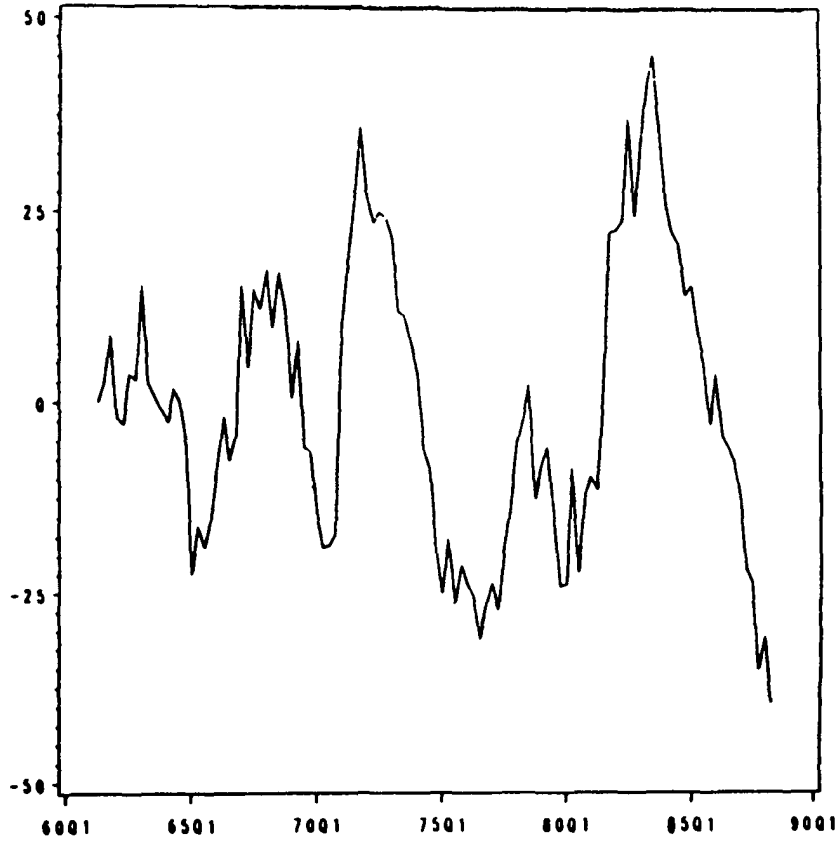


Fig. 4. Swedish Detrended Unemployment. Deviation from Trend.

The industrial production series is detrended by

$$IP_t = a_2 + \beta_2 t + \beta_3(t-57) DUM + \epsilon_t; \quad (1)$$

$$\text{where } DUM = \begin{cases} 1 & \text{if } t > 57 \\ 0 & \text{otherwise} \end{cases}$$

( $t=57$  corresponds with observation 74:II), while the corresponding trend in unemployment  $U_t$  is modeled by

$$U_t = a_3 + \beta_4 t + \epsilon_t; \quad (2)$$

The distributions of changes of industrial production and unemployment are illustrated by Figures 5-6.

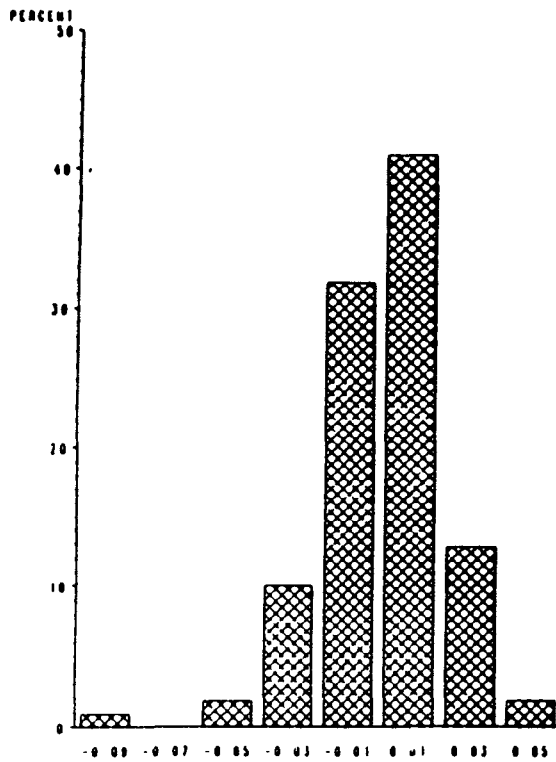


Fig. 5. Swedish Industrial Production; Distribution of Changes in Detrended, Seasonally Adjusted Data; 60:I-88:I.

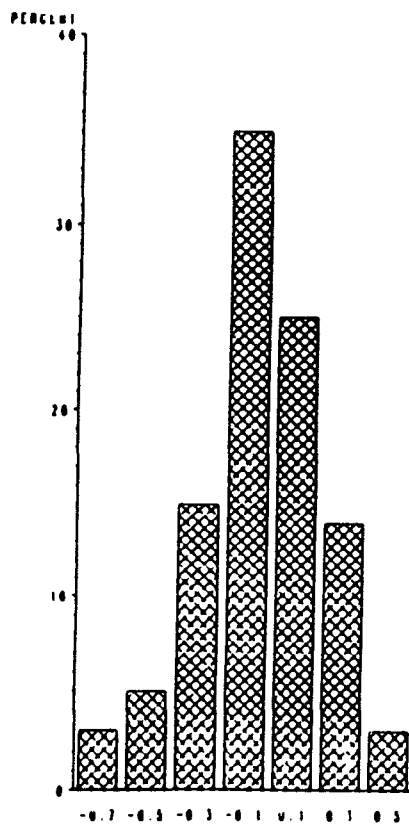


Fig. 6. Swedish Unemployment; Distribution of Changes in Detrended, Seasonally Adjusted Data; 60:I-88:I.

## 2.2 The Markov Process Test Procedure

As indicated in the previous section, the present test procedure focuses on sequences of differences in  $y_t$  ( $t = 1, \dots, T$ ), the seasonal and trend adjusted quarterly time series. It is assumed that  $\{y_t(\omega), t=0,1,\dots; \omega \in \Omega\}$  is a stationary and regular stochastic process defined on a probability space  $(\Omega, F, P)$  where  $\omega \in \Omega$  are events defined by particular realizations of the process  $y_t$ . Denote the differences by  $\Delta y_t$ , and introduce an indicator process  $\{D_t\}$ , such that

$$D_t = \begin{cases} 1, & \text{when } \Delta y_t > 0 \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

Although  $y_t$  is perhaps not fully stationary,  $D_t$  will probably be so (which is a necessary assumption for the test procedure).

For quarterly data it is reasonable to assume that  $\{D_t\}$  follows a second-order finite Markov stochastic process. The asymmetry concept introduced above then implies that the transition probability

$$\lambda_{ii} = P(D_t = i \mid D_{t-1} = D_{t-2} = i) \quad (4)$$

differs significantly between  $i = 1$  and  $i = 0$ .

However, this is only one of several symmetry aspects of the business cycle. Symmetry could of course be defined according to other sequences and the transition probabilities of these sequences tested. For instance, the events  $\{\Delta y_t > 0 \mid \Delta y_{t-1} \leq 0, \Delta y_{t-2} \leq 0\}$  and  $\{\Delta y_t > 0 \mid \Delta y_{t-1} > 0, \Delta y_{t-2} \leq 0\}$  could be associated with the shift from recession to expansion. The corresponding shift from expansion to recession could then be defined as  $\{\Delta y_t \leq 0 \mid \Delta y_{t-1} \leq 0, \Delta y_{t-2} > 0\}$  and  $\{\Delta y_t \leq 0 \mid \Delta y_{t-1} > 0, \Delta y_{t-2} > 0\}$ . This aspect of symmetry has also been considered in the paper by testing the hypothesis  $H_0: \lambda_{01} = \lambda_{10}$ . This test focuses the interest on the behavior of the business cycle in times of switching

from recession to expansion and vice versa. This test is of interest per se, because the time when the business cycle turns is of special importance for understanding the dynamics of the whole business cycle.

The Markov process test is restricted to the case where  $\{y_t\}$  is a stationary and ergodic process with mean zero. This implies that trends in  $\{y_t\}$  must be removed before the test can be used. Falk has examined several approaches for trend removal and concluded that the test is not sensitive to the modeling of the trend component. Of course, this could not be taken for granted. Let  $\mathfrak{R}$  be a class of trends and  $\{\bar{y}_t\}$  a process defined by

$$\{\bar{y}_t\} = \{y_t\} + \mathfrak{R}(t)$$

Then  $\{y_t\} = \{\bar{y}_t\} - \mathfrak{R}(t)$  would not contain a trend and  $\{y_t\}$  could be used. Falk claims that if several members of  $\mathfrak{R}$  will provide the same distribution of  $N_{ij}$  (where  $N_{ij}$  denotes the observed frequency of sequences  $\{D_i, D_j\}$ ) given the realization  $S_T = \{i_1, i_2, \dots, i_T\}$ , then the properties of the test do not depend on  $\mathfrak{R}$ . Now, even if every member of  $\mathfrak{R}$  will produce identical distributions of  $N_{ij}$ , yet trend removal is not irrelevant.

Let us assume that  $\{y_t\} = \{y^*_t\} + \{v_t\}$  where  $\{y^*_t\}$  is symmetric while  $\{v_t\}$  is not symmetric in the sense of  $\lambda_{00} \neq \lambda_{11}$ . It is possible that the trend removal procedure also eliminates the asymmetry component  $\{v_t\}$ . This seems to be of special significance when we allow  $\mathfrak{R}$  to be stochastic in the sense of Harvey (1985).

Now, because  $D_t$  is invariant to every monotone transformation of  $\Delta y_t$  we might trust the robustness of the procedure with respect to trend elimination.

The transition probabilities in (4) can now be estimated by maximizing the likelihood function

$$\begin{aligned} L(S_T) &= P(D_1=i_1, D_2=i_2)P(D_3=i_3|D_1=i_1, D_2=i_2), \dots, P(D_T=i_T|D_{T-2}=i_{T-2}, D_{T-1}=i_{T-1}); \\ &= \pi_0(\lambda_{ij}) \prod_i \prod_j \lambda_{ij}^{N_{ij}} (1-\lambda_{ij})^{T_{ij}}; \quad i, j = 0, 1 \end{aligned} \quad (5)$$



where  $\pi_0$  is the probability of the first two realizations of  $D_t$   $\{D_1=i_1, D_2=i_2\}$  in the sample.  $\lambda_{ij}, (1-\lambda_{ij})$  are transition probabilities with the corresponding sample frequencies  $N_{ij}$  and  $T_{ij}$ . The likelihood function  $L$  is nonlinear in  $\lambda_{ij}$  and so will the derivatives be because  $\pi_0$  is depending on  $\lambda_{ij}$  for every initial state  $\{0,0\}, \{0,1\}, \{1,0\}$ , and  $\{1,1\}$  of  $D_1$  and  $D_2$ . There are in fact four likelihood functions depending on which initial state is observed in the sample. If  $\{D_t\}$  is stationary and regular, the probabilities  $\pi_0=\pi_{ij}$  can be expressed in terms of the transition probabilities  $\lambda_{ij}$ , (see Appendix 1).

Let  $\hat{\lambda}$  be the maximum likelihood estimate of  $\lambda_{ij}$  while the matrix of second order partial derivatives of  $\log L(S_T)$  is denoted by  $H$ . Then, it is well known that  $-H^{-1}$  is an asymptotic estimate of the covariance matrix of  $\lambda_{ij}$  in the point of  $L(S_T)_{\max}$  and that

$$[\lambda \quad \hat{\lambda}]' (-H^{-1}) [\lambda \quad \hat{\lambda}] = \chi_k^2(a) \quad (6)$$

is a  $a$  percent confidence region of  $\lambda$  with  $k$  degrees of freedom where  $k$  is the dimension of  $H$ . Linear hypotheses of  $\lambda$  could be tested by the use of (6) above. For example if the region  $R$  defined by the hypothesis  $\lambda=\lambda_0$  and the interior of the confidence region (6) intersect, we cannot reject the hypothesis  $\lambda=\lambda_0$ . We are following Neftçi and Falk by testing the hypothesis

$$H_0: \lambda_{11} = \lambda_{00}$$

of symmetry between upswings and downswings.

Neftçi and Falk are considering the test  $H_0: \lambda_{11} = \lambda_{00}$  as a test of similarity of upswings and downswings, i.e. sequences of the type

$$\{\Delta y_t > 0 | \Delta y_{t-1} > 0, \Delta y_{t-2} > 0\} \text{ and } \{\Delta y_t \leq 0 | \Delta y_{t-1} \leq 0, \Delta y_{t-2} \leq 0\}$$

are equally likely. It should be pointed out that  $H_0$  above is equivalent to  $1 - \lambda_{11} = 1 - \lambda_{00}$  which means that the sequences

$$\{\Delta y_t \leq 0 | \Delta y_{t-1} > 0, \Delta y_{t-2} > 0\} \text{ and } \{\Delta y_t > 0 | \Delta y_{t-1} \leq 0, \Delta y_{t-2} \leq 0\}$$

also must be equally likely. This can be seen by the definitions of transition probabilities in Appendix 1. Consequently  $H_0$  above does in fact capture phases of expansions and recessions and partly switches from expansion to recession and vice versa (switches from recession to expansion and from expansion to recession can occur in other ways). However, it must be made clear that the acceptance of  $H_0$  is not equivalent to stating that  $\pi_{00} = \pi_{11}$ , i.e. that the occurrences of

$$\{\Delta y_t > 0, \Delta y_{t-1} > 0\} \text{ and } \{\Delta y_t \leq 0, \Delta y_{t-1} \leq 0\}$$

are equally likely, except if transition probabilities  $\lambda_{01}$  and  $\lambda_{10}$  are equal. This can be seen in Appendix 1. For this reason the concept of symmetry has been expanded by testing for symmetry between expansion and recession, and between recession and expansion by

$$H_0: \lambda_{01} = \lambda_{10}.$$

### 2.3 Estimating the Transition Probabilities

Due to the nonlinearities of  $L$ , maximum likelihood estimates of  $\lambda$  cannot be found analytically. By finding the maximum of  $\log L$  with respect to  $\lambda_{ij}$  given the sample  $S_T$ , we could apply Newton's iterative algorithm, which uses the derivatives of  $\log L$ . These calculations will be found in Appendix 2 for the initial probabilities  $\pi_0 = \pi_{00}$ ,  $\pi_0 = \pi_{01} = \pi_{10}$  and  $\pi_0 = \pi_{11}$ . The estimation program was coded in SAS/IML matrix code and it uses the

first and second analytic derivatives of the likelihood function (see Appendix 2). The linear part of the likelihood function was used to form the approximate estimates of the transition probabilities.<sup>1</sup> As a matter of fact the initial estimates were very close to the final ML-estimates. This is of course dependent on the available amount of information on the number of sequences. For the Swedish data there were 114 sequences of  $D_t$ .

The validation of the SAS program was done in two steps. First a program not using analytical derivatives was used and the estimates were compared. Then the transition probabilities for some international series published by Falk were estimated and compared to Falk's results.<sup>2</sup> The convergence criterion for the algorithm was set to the order  $10^{-14}$ .

In estimating the transition probabilities  $\lambda_{00}$ ,  $\lambda_{11}$ ,  $\lambda_{10}$  and  $\lambda_{01}$  the likelihood function (5) was formed for the relevant initial states  $D_1$  and  $D_2$ . Then the number of observed sequences  $N_{ij}$  and  $T_{ij}$  was used for approximate estimates of the transition probabilities. The point estimates  $\lambda_{00}$  and  $\lambda_{11}$  and the relevant part of the Hessian matrix were then used to form 80 % confidence regions for  $\lambda_{00}$  and  $\lambda_{11}$ . If the confidence region and the line  $\lambda_{00}=\lambda_{11}$  intersected, we would not be able to reject the hypothesis of symmetry in the sense that the transition probabilities of expansions and contractions are the same.

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<sup>1</sup> This means that the initial probability  $\pi_0$  is ignored which results in the solution  $\hat{\lambda}_{ij} = (N_{ij}/(N_{ij}+T_{ij}); i,j = 0,1$ .

<sup>2</sup> The estimates were quite close, with the exception of the one for the United Kingdom. It seems that Falk has used the initial probability  $\pi_0 = \pi_{00}$  instead of the correct  $\pi_0 = \pi_{10}$ .

## 2.4 Some Test Results

### Industrial Production

In Table 1 below the sequences  $N_{ij}$  and  $T_{ij}$  forming the likelihood function are shown for Canada, France, Italy, United Kingdom, West Germany and Sweden. The data cover the period 1960:1–1988:1 for Sweden and 1951:1–1983:4 for the other countries.<sup>3</sup>

TABLE 1. Summary of Sequences  $N_{ij}$  and  $T_{ij}$  for Industrial Production

	Canada	France	Italy	U.K.	Sweden	West Germany
$N_{00}$	24	28	22	26	25	10
$T_{00}$	16	11	17	14	12	10
$N_{11}$	24	18	21	17	27	19
$T_{11}$	15	14	15	13	16	14
$N_{10}$	16	13	15	14	16	14
$T_{10}$	9	16	12	16	8	15
$N_{01}$	16	11	17	13	12	10
$T_{01}$	9	18	10	16	13	18
$D_1, D_2$	1,0	1,1	1,0	1,0	1,0	1,0

<sup>3</sup>  $N_{ij}$  and  $T_{ij}$  are partly taken from Table 3 in Falk (1986).

TABLE 2. Transition Probabilities for Detrended International Industrial  
Production

	Canada	France	Italy	U.K.	Sweden	West Germany
$\lambda_{00}$	.596 (.077)	.713 (.072)	.560 (.079)	.645 (.075)	.671 (.077)	.495 (.111)
$\lambda_{11}$	.610 (.077)	.576 (.088)	.579 (.082)	.562 (.090)	.623 (.073)	.571 (.085)
$\lambda_{01}$	.635 (.097)	.373 (.090)	.625 (.093)	.442 (.092)	.474 (.100)	.353 (.090)
$\lambda_{10}$	.635 (.097)	.462 (.091)	.550 (.096)	.462 (.091)	.662 (.097)	.478 (.093)
$-\partial^2 L / \partial \lambda_{00}^2$	168.60	194.37	160.57	177.67	170.62	81.33
$-\partial^2 L / \partial \lambda_{11}^2$	166.74	128.99	149.94	123.84	186.09	137.01
$-\partial^2 L / \partial \lambda_{00} \partial \lambda_{11}$	-.601	-.603	-.448	.458	.754	.247
$-\partial^2 L / \partial \lambda_{01}^2$	107.16	124.24	114.51	117.42	100.05	123.01
$-\partial^2 L / \partial \lambda_{10}^2$	107.10	120.56	108.67	120.57	106.25	115.97
$-\partial^2 L / \partial \lambda_{01} \partial \lambda_{10}$	-.234	-.426	-.241	-.348	-.298	-.318

H<sub>0</sub>:  $\lambda_{00} = \lambda_{11}$

The transition probabilities  $\lambda_{11}$  for the six countries are surprisingly close to each other varying from 0.56 to 0.62. There is greater variation in the transition probabilities  $\lambda_{00}$ . This is apparent from Table 2 above, and also from Fig. 7, where the 80 % confidence regions of  $\lambda_{00}$  and  $\lambda_{11}$  for the six countries are graphed. The 45 degree line intersects every confidence region, which means that the hypothesis of equal transition probabilities of expansion and recession cannot be rejected at the 20 % significance level.  $\lambda_{00}$  and  $\lambda_{11}$  are most asymmetric in France with a difference of 0.137.

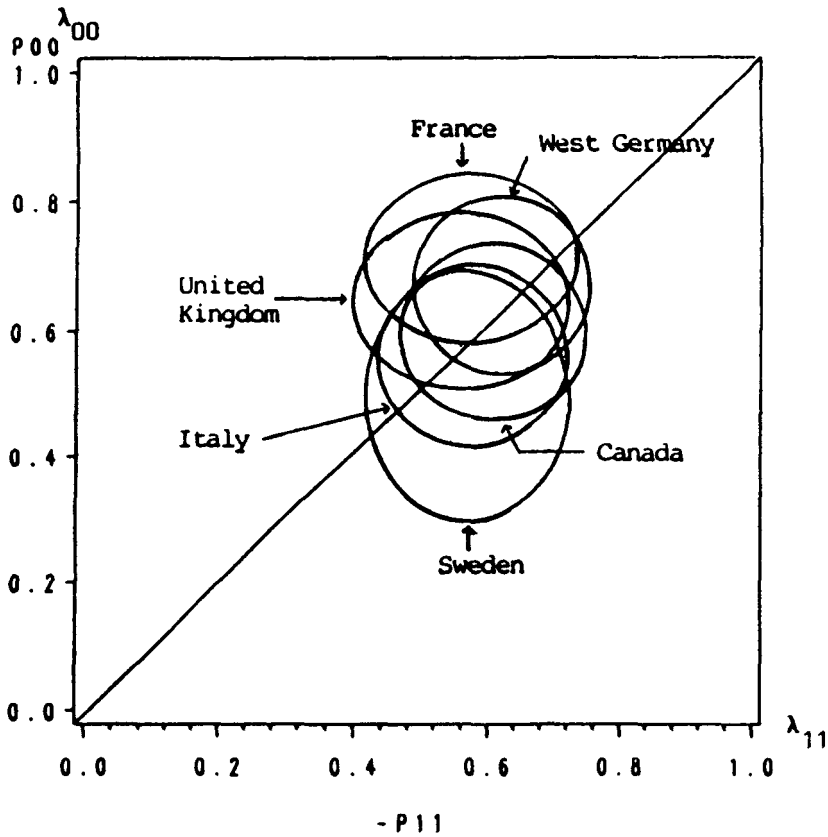


Fig. 7 80 % Confidence Region for Transition Probabilities  $\lambda_{00}$  and  $\lambda_{11}$  in Industrial Production.

$H_0: \lambda_{01} = \lambda_{10}$

This hypothesis implies that the transition probabilities of a switch from expansion to recession and from recession to expansion are the same. The 80 % confidence regions for  $\lambda_{01}$  and  $\lambda_{10}$  are shown in Fig. 8. There seems to be greater variation in these transition probabilities than in the corresponding probabilities for  $\lambda_{00}$ . The regions for Canada and Sweden differ almost significantly, as do those for Canada and France. As the 45 degree line intersects every confidence region, the conclusion will be the same as in the case of  $H_0: \lambda_{00} = \lambda_{11}$ ; i.e. there is no significant difference between  $\lambda_{01}$  and  $\lambda_{10}$  for the six countries studied. West Germany seems to be closest to the rejection of the hypothesis of symmetry between the transition from expansion to recession and from recession to expansion.

In conclusion we have to state that in the case of industrial production there is no significant support for the existence of asymmetry in the two aspects discussed.

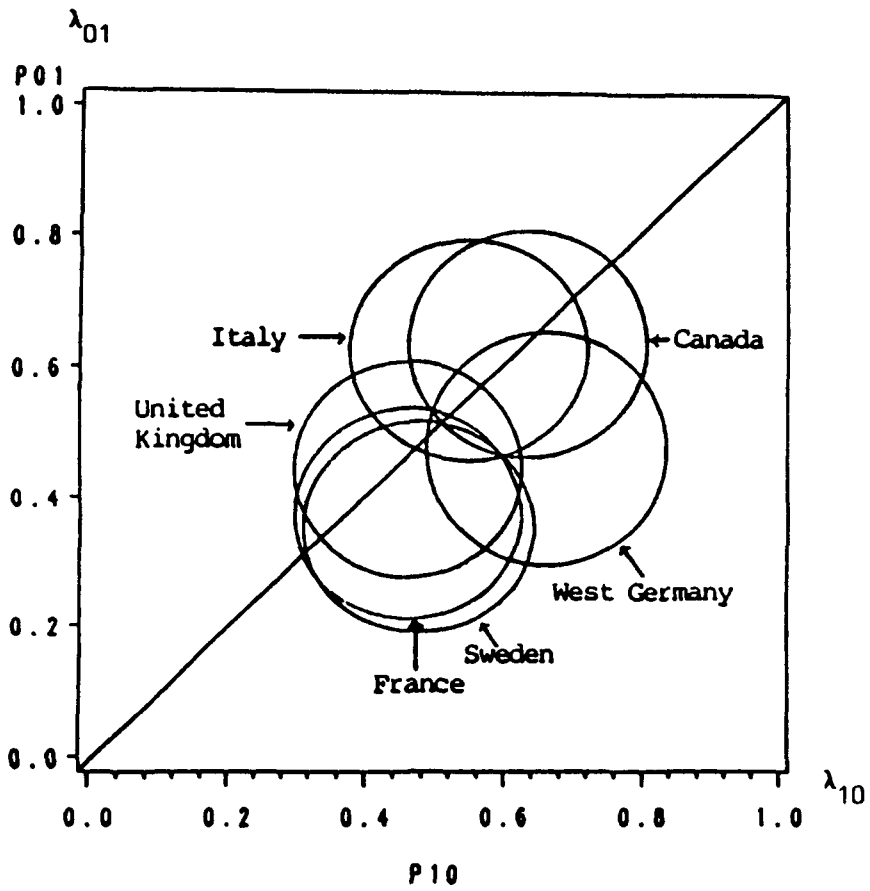


Fig. 8. 80 % Confidence Region for Transition Probabilities  $\lambda_{01}$  and  $\lambda_{10}$  in Industrial Production.

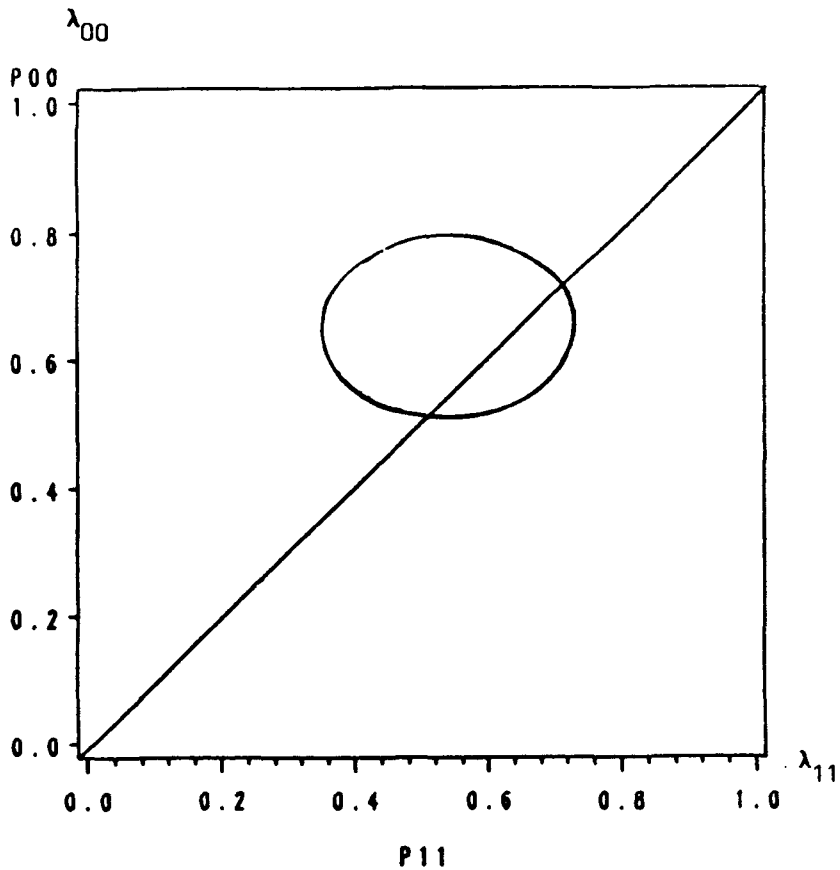


Fig. 9. 80 % Confidence Region for Transition Probabilities  $\lambda_{00}$  and  $\lambda_{11}$  in Swedish Unemployment.

Swedish Unemployment

Table 3 below shows the sequences  $N_{ij}$  and  $T_{ij}$  for Swedish unemployment.

TABLE 3. Sequences  $N_{ij}$ ,  $T_{ij}$  and ML-estimates for Swedish Unemployment

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$N_{00} = 23$	$\lambda_{00} = 0.651$	$-\partial^2 L / \partial \lambda_{00}^2 = 157.3$
$T_{00} = 12$	(0.080)	
$N_{11} = 12$	$\lambda_{11} = 0.540$	$-\partial^2 L / \partial \lambda_{11}^2 = 90.25$
$T_{11} = 11$	(0.102)	
$N_{10} = 10$	$\lambda_{10} = 0.435$	$-\partial^2 L / \partial \lambda_{10}^2 = -0.439$
$T_{10} = 14$	(0.099)	
$N_{01} = 12$	$\lambda_{01} = 0.493$	
$T_{01} = 12$	(0.105)	

---

| $D_1, D_2 = 1, 1$ |  |  |

---

There is some evidence of asymmetry. The transition from decrease to decrease seems to last longer than the transition from increase to increase. However, in Fig. 9 every 80 % confidence region and the 45 degree line intersect. Therefore the hypothesis  $H_0: \lambda_{00} = \lambda_{11}$  cannot be rejected at 20 % confidence level.



### 3. Testing Swedish Business Cycle Asymmetry by Skewness Statistics

Although the procedure outlined in Chapter 2 has not so far been thoroughly analyzed with respect to its power properties, its power is expected to be rather weak. Basically, that assumption originates in the fact that the time series used are transformed and that the information is reduced by the dichotomization of  $\{\Delta y_t\}$  into  $\{D_t\}$ . The approach used by Neftçi was criticized by DeLong and Summers (1984) from that point of view. They proposed that an ordinary test of skewness of the distribution of relative changes in the key variable should be used:

$$S(\Delta y_t) = \frac{m_3(\Delta y_t)}{[m(\Delta y_t)]^3} \quad (7)$$

where  $m_3(\Delta y_t)$  = the 3rd centered moment of  $\Delta y_t$ , and  $m(\Delta y_t)$  = the standard deviation of  $\Delta y_t$ .

In the case of symmetry (i.e. for  $S(\Delta y_t) = 0$ ), the estimated skewness is assumed to be normally distributed:

$$\hat{S}(\Delta y_t) \overset{\text{approx}}{\sim} N\left[0, \frac{6}{T}\right] \quad (8)$$

The variance in (8) requires independent observations, and as  $\Delta y_t$  is a series with significant serial correlation, the variance needs adjustment. As it is a function of the serial correlation, it must be approximately determined through simulations. A time series model for  $\Delta y_t$  is first verified, where the residuals are normally distributed "white noise". A number of artificial time series are then generated by adding normally distributed random numbers to the estimated model. The skewness is determined for each replicate of the series, and finally the standard deviation of the skewness is estimated over all the replicates. This standard deviation estimate is then used in the test procedure.

DeLong and Summers used that procedure to test for asymmetry in key variables such as GNP and unemployment in the US and in some other countries. With the exception of unemployment in the US, they could not find any empirical support for the existence of asymmetries in business cycle variables. They used an AR(3) process with normally distributed errors  $\epsilon_t$  to model the serial correlation in  $\Delta y_t$ :

$$\Delta y_t = \alpha_0 + \sum_{j=1}^3 \alpha_j \Delta y_{t-j} + \epsilon_t \quad (t = 1, \dots, T) \quad (9)$$

We used the same approach, although an AR(2) process seemed a possible alternative, at least in the case of the industrial production data. The estimated autoregressive models are shown in Table 4 below.

TABLE 4. Estimated autoregressive model for  $\Delta y_t$  (the estimated standard deviation is given in parentheses).

$\Delta y_t$	Intercept ( $\hat{\alpha}_0$ )	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$R^2$
Industrial production	0.007 (0.002)	0.714 (0.097)	0.296 (0.119)	-0.097 (0.100)	0.795
Unemployment	-386.57 (771.18)	0.899 (0.097)	0.219 (0.131)	(-0.216) (0.009)	0.821

The results we obtained by using (7) as test statistics on Swedish data are summarized in Table 5. They are very similar to those obtained by the Markov process testing in Section 2. Above all, they show similarities with corresponding test results for West German data (cf. Table 2 in DeLong and Summers, op.cit.).

TABLE 5. Estimated Skewness in Differenced Industrial Production and Unemployment in Sweden (1960:I – 1988:I).

Variable	Skewness	Standard Deviation	Min.	Max.
Industrial production	–0.048	0.243	–0.706	0.523
Unemployment	0.119	0.203	–0.355	0.570

#### 4. Conclusions

This study is treating several aspects of asymmetry in business cycles in Sweden, Canada, France, Italy, the United Kingdom and West Germany. It is part of a project termed EWSIS (Early Warning System in Sweden), financed by Statistics Sweden. One of the purposes of this project is to study and evaluate the properties of some of the leading indicators for Swedish business cycles used in the OECD system of leading indicators. Industrial production and unemployment have been selected as key variables for monitoring the business cycles. Ideally, the research project will result in a forecasting system for Swedish business cycles, partly based on leading indicators. The idea of a leading indicator is that major events (e.g. peaks) of a reference series will be signalled well in advance by the leading indicator. For EWSIS the reference point has been the OECD system of leading indicators, with a special interest in indicators with long leads. However, there are several important differences between the EWSIS and the OECD system. The most essential one refers to problems of structural variability. One example of structural variability is variation in lead time between upswings and downswings for a leading indicator of business cycles. This is a severe complication with special reference

to economic forecasting. In developing EWSIS special consideration has been made to the existence and implications of structural variability.

Because industrial production is the key variable in EWSIS, asymmetries in that series must be considered for the reasons stated above. Furthermore, as Sweden is an open economy with great dependence on foreign markets, this dependence must be taken into consideration. Consequently, leads between Sweden and other countries with respect to industrial production are studied. If this information is to be used in a formalized forecasting system, it is of course of special interest to ascertain whether these foreign series exhibit asymmetries as well. For that reason, some potentially interesting international series have been included in the study.

Asymmetry can be defined and handled in several ways. In this paper we have partly employed the approach used by Neftçi (1984) and by Falk (1986), where the probability of consecutive changes of direction  $\Delta y_t$  of a series  $y_t$  is studied in the framework of Markov processes of the second order. If  $\lambda_{11}$  is the transition probability that the change  $\Delta y_t$  is positive, conditional on the event  $\{\Delta y_{t-1} > 0, \Delta y_{t-2} > 0\}$ , i.e. if  $\lambda_{11}$  is the probability that the business cycle will remain in 'expansion', while the cycle was in 'expansion' during two periods before and if  $\lambda_{00}$  is the corresponding probability for recession defined by  $\lambda_{00} = P\{\Delta y_t \leq 0 | \Delta y_{t-1} \leq 0, \Delta y_{t-2} \leq 0\}$ , then the likelihood principle could be used to test the hypothesis  $\lambda_{00} = \lambda_{11}$  for symmetry of expansions and contractions of the business cycle. If this hypothesis cannot be rejected, the probability of remaining in expansion is the same as the probability of remaining in recession. If, on the other hand, the test supports the hypothesis  $\lambda_{00} \neq \lambda_{11}$ , we must conclude that there is asymmetry in the series  $\Delta y_t$ . The test  $H_0: \lambda_{00} = \lambda_{11}$  is equivalent with  $H_0: 1 - \lambda_{00} = 1 - \lambda_{11}$ , which means that the transition probability from states  $\{\Delta y_{t-1} > 0, \Delta y_{t-2} > 0\}$  to  $\{\Delta y_t \leq 0\}$  and from  $\{\Delta y_{t-1} \leq 0, \Delta y_{t-2} \leq 0\}$  to  $\{\Delta y_t > 0\}$  must also be equal. Beside these aspects of symmetry, other switches from recession to expansion and vice versa have been considered by means of testing  $H_0: \lambda_{01} = \lambda_{10}$ .

The transition probabilities have been estimated by means of the likelihood principle. The likelihood function, which is nonlinear in the transition parameters has been maximized using an iterative algorithm of the Newton type, using first and second order partial derivatives of the likelihood function. The algorithm is coded in SAS/IML language.

The results confirm those shown by Falk. There is no significant empirical evidence of asymmetry in the business cycles studied in any country. The most asymmetric series are the unemployment series, especially for the United States, as reported by Neftçi. However, the asymmetry of Swedish unemployment is small and not significant on the 20 % confidence level. These results are further confirmed by tests for asymmetry based on skewness statistics.

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## Appendix 1 . Derivation of the Stationary Distribution $\pi_{ij}$

### Notations

$$\begin{aligned}
 \pi_{00}(t) &= P(D_t = -1, D_{t-1} = -1) & \pi_{11}(t) &= P(D_t = 1, D_{t-1} = 1) \\
 \pi_{10}(t) &= P(D_t = 1, D_{t-1} = -1) & \pi_{01}(t) &= P(D_t = -1, D_{t-1} = 1) \\
 \lambda_{00} &= P(D_t = -1, | D_{t-1} = -1, D_{t-2} = -1); & \lambda_{11} &= P(D_t = 1, | D_{t-1} = 1, D_{t-2} = 1); \\
 \lambda_{10} &= P(D_t = 1, | D_{t-1} = 1, D_{t-2} = -1); & \lambda_{01} &= P(D_t = -1, | D_{t-1} = -1, D_{t-2} = 1); \\
 (1-\lambda_{00}) &= P(D_t = 1, | D_{t-1} = -1, D_{t-2} = -1); & (1-\lambda_{11}) &= P(D_t = -1, | D_{t-1} = 1, D_{t-2} = 1); \\
 (1-\lambda_{10}) &= P(D_t = -1, | D_{t-1} = 1, D_{t-2} = -1); & (1-\lambda_{01}) &= P(D_t = 1, | D_{t-1} = -1, D_{t-2} = 1);
 \end{aligned}$$

By making use of a) – d) below where

- a)  $\{D_t=1, D_{t-1}=1, D_{t-2}=1\}$  and  $\{D_t=1, D_{t-1}=1, D_{t-2}=-1\}$  are disjoint
- b)  $\{D_t=1, D_{t-1}=1\} = \{D_t=1, D_{t-1}=1, D_{t-2}=1\} \cup \{D_t=1, D_{t-1}=1, D_{t-2}=-1\}$
- c)  $P(D_t=1, | D_{t-1}=1, D_{t-2}=1) = \frac{P(D_t=1, D_{t-1}=1, D_{t-2}=1)}{P(D_{t-1}=1, D_{t-2}=1)}$
- d)  $\sum \pi_{ij} = 1$

the expression for  $\pi_{ij}$  in terms of  $\lambda_{ij}$  will be as follows:

$$\begin{aligned}
 \pi_{11}(t) &= \lambda_{11} \pi_{11}(t-1) + \lambda_{10} \pi_{10}(t-1); & \pi_{00}(t) &= \lambda_{00} \pi_{00}(t-1) + \lambda_{01} \pi_{01}(t-1); \\
 \pi_{10}(t) &= (1-\lambda_{01}) \pi_{01}(t-1) + (1-\lambda_{00}) \pi_{00}(t-1); & \pi_{01}(t) &= (1-\lambda_{11}) \pi_{11}(t-1) + (1-\lambda_{10}) \pi_{10}(t-1);
 \end{aligned}$$

If the process  $\{D_t\}$  is stationary the distribution  $\pi_{ij}(t)$  will not depend on  $t$ , i.e.,

$\pi_{ij}(t) = \pi_{ij}(t-1) = \pi_{ij}$ . The equation system could then be written

$$\begin{aligned}
 \pi_{11} &= \lambda_{11} \pi_{11} + \lambda_{10} \pi_{10}; & \pi_{00} &= \lambda_{00} \pi_{00} + \lambda_{01} \pi_{01}; \\
 \pi_{10} &= (1-\lambda_{01}) \pi_{01} + (1-\lambda_{00}) \pi_{00}; & \pi_{01} &= (1-\lambda_{11}) \pi_{11} + (1-\lambda_{10}) \pi_{10}
 \end{aligned}$$

with the solution

$$\pi_{01} = \pi_{10} = \delta_{00} \delta_{11} D^{-1};^1 \quad \pi_{00} = \lambda_{01} \delta_{11} D^{-1}; \quad \pi_{11} = \lambda_{10} \delta_{00} D^{-1};$$

where  $D$  and  $\delta_{ij}$  is defined in Appendix 2.

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<sup>1</sup> This result is identical with the corresponding one shown in Falk (1986, p. 1098). Neftçi (1984, p. 326–327) has arrived at a different solution for  $\pi_{01}$  and  $\pi_{10}$ .

## Appendix 2. Derivatives of log L

The likelihood function of the sample ( $S_T$ ) is given by

$$L(\pi_0, \lambda_{ij} | S_T) = \pi_0 \prod_i \prod_j \lambda_{ij}^{N_{ij}} (1 - \lambda_{ij})^{T_{ij}} ; \quad i, j = 0, 1$$

where  $\pi_0$  is the probability of the initial state  $\{D_2=i_2, D_1=i_1\}$ . There are four possible initial states  $\{D_2=1, D_1=1\}$ ,  $\{D_2=1, D_1=-1\}$ ,  $\{D_2=-1, D_1=1\}$  and  $\{D_2=-1, D_1=-1\}$  and the proper selection of  $\pi_0$  depends on the first two outcomes of  $D_1$  and  $D_2$ . Neftçi (1984) provides some of the derivatives of log L for the case  $\pi_0 = \pi_{00}$ , corresponding to an initial sequence of  $\{D_2=-1, D_1=-1\}$ . For the data used in this study, starting from 60:I, the first sequence of the sample is  $\{D_2=1, D_1=-1\}$  and the relevant  $\pi_0$  would then be  $\pi_{01}$ . Below we show the derivatives of log L for  $\pi_0=\pi_{00}$  and  $\pi_0=\pi_{01}=\pi_{10}$  and  $\pi_0=\pi_{11}$ . These derivatives are shown in detail because there seems to be some misprints in Neftçi. Some of them were also noticed by Falk (1986). The following substitutions will be used:

$$\begin{aligned} \delta_{ij} &= (1 - \lambda_{ij}); & \varphi_0 &= 2(1 - \lambda_{00}) + \lambda_{01}; & \varphi_1 &= 2(1 - \lambda_{11}) + \lambda_{10}; \\ K1_{ij} &= N_{ij} \lambda_{ij}^{-1} - T_{ij} \delta_{ij}^{-1}; & K2_{ij} &= N_{ij} \lambda_{ij}^{-1} + T_{ij} \delta_{ij}^{-1}; & L &= \log L; \\ D &= \delta_{00}(\delta_{11} + \lambda_{10}) + \delta_{11}(\delta_{00} + \lambda_{01}); & i, j &= 0, 1 \end{aligned}$$

$$\pi_0 = \pi_{00}$$

$$\begin{aligned} \partial L / \partial \lambda_{00} &= \varphi_1 D^{-1} + K1_{00}; & \partial L / \partial \lambda_{01} &= \lambda_{01}^{-1} - \delta_{11} D^{-1} + K1_{01}; \\ \partial L / \partial \lambda_{10} &= -\delta_{00} D^{-1} + K1_{10}; & \partial L / \partial \lambda_{11} &= \varphi_0 D^{-1} - \delta_{11}^{-1} + K1_{11}; \end{aligned}$$



The second partial derivatives of  $\log L$  will be given by

$$\begin{aligned}
 \partial^2 L / \partial \lambda_{00}^2 &= \varphi_1^2 D^{-2} - K_{200}; & \partial^2 L / \partial \lambda_{00} \partial \lambda_{01} &= -\varphi_1 \delta_{11} D^{-2}; \\
 \partial^2 L / \partial \lambda_{00} \partial \lambda_{10} &= -\varphi_1 \delta_{00} D^{-2} + D^{-1}; & \partial^2 L / \partial \lambda_{00} \partial \lambda_{11} &= \varphi_0 \varphi_1 D^{-2} - 2 D^{-1}; \\
 \partial^2 L / \partial \lambda_{01}^2 &= -\lambda_{01}^{-2} + \delta_{11}^2 D^{-2} - K_{201}; & \partial^2 L / \partial \lambda_{01} \partial \lambda_{10} &= \delta_{00} \delta_{11} D^{-2}; \\
 \partial^2 L / \partial \lambda_{01} \partial \lambda_{11} &= D^{-1} - \varphi_0 \delta_{11} D^{-2}; & \partial^2 L / \partial \lambda_{10}^2 &= \delta_{00}^2 D^{-2} - K_{210}; \\
 \partial^2 L / \partial \lambda_{10} \partial \lambda_{11} &= -\varphi_0 \delta_{00} D^{-2}; & \partial^2 L / \partial \lambda_{11}^2 &= \varphi_0^2 D^{-2} - \delta_{11}^{-2} - K_{111};
 \end{aligned}$$

$$\pi_0 = \pi_{01} = \pi_{10}$$

$$\begin{aligned}
 \left. \frac{\partial L}{\partial \lambda_{00}} \right|_{\pi_0=\pi_{01}=\pi_{10}} &= \left. \frac{\partial L}{\partial \lambda_{00}} \right|_{\pi_0=\pi_{00}} - \delta_{00}^{-1}; & \left. \frac{\partial L}{\partial \lambda_{01}} \right|_{\pi_0=\pi_{01}=\pi_{10}} &= \left. \frac{\partial L}{\partial \lambda_{00}} \right|_{\pi_0=\pi_{00}} - \lambda_{01}^{-1}; \\
 \left. \frac{\partial^2 L}{\partial \lambda_{00}^2} \right|_{\pi_0=\pi_{01}=\pi_{10}} &= \left. \frac{\partial^2 L}{\partial \lambda_{00}^2} \right|_{\pi_0=\pi_{00}} - \delta_{00}^{-2}; & \left. \frac{\partial^2 \log L}{\partial \lambda_{01}^2} \right|_{\pi_0=\pi_{01}=\pi_{10}} &= \left. \frac{\partial^2 \log L}{\partial \lambda_{01}^2} \right|_{\pi_0=\pi_{00}} + \lambda_{01}^{-2};
 \end{aligned}$$

The other partial derivatives will be identical to  $\pi_0 = \pi_{00}$  and  $\pi_0 = \pi_{01}$

$$\pi_0 = \pi_{11}$$

$$\begin{aligned}
 \left. \frac{\partial L}{\partial \lambda_{00}} \right|_{\pi_0=\pi_{11}} &= \left. \frac{\partial L}{\partial \lambda_{00}} \right|_{\pi_0=\pi_{00}} - \delta_{00}^{-1}; & \left. \frac{\partial L}{\partial \lambda_{01}} \right|_{\pi_0=\pi_{11}} &= \left. \frac{\partial L}{\partial \lambda_{01}} \right|_{\pi_0=\pi_{00}} - \lambda_{01}^{-1}; \\
 \left. \frac{\partial L}{\partial \lambda_{10}} \right|_{\pi_0=\pi_{11}} &= \left. \frac{\partial L}{\partial \lambda_{10}} \right|_{\pi_0=\pi_{00}} + \lambda_{10}^{-1}; & \left. \frac{\partial L}{\partial \lambda_{11}} \right|_{\pi_0=\pi_{11}} &= \left. \frac{\partial L}{\partial \lambda_{11}} \right|_{\pi_0=\pi_{00}} + \delta_{11}^{-1}; \\
 \left. \frac{\partial^2 L}{\partial \lambda_{00}^2} \right|_{\pi_0=\pi_{11}} &= \left. \frac{\partial^2 L}{\partial \lambda_{00}^2} \right|_{\pi_0=\pi_{00}} - \delta_{00}^{-2}; & \left. \frac{\partial^2 L}{\partial \lambda_{10}^2} \right|_{\pi_0=\pi_{11}} &= \left. \frac{\partial^2 L}{\partial \lambda_{10}^2} \right|_{\pi_0=\pi_{00}} - \lambda_{10}^{-2}; \\
 \left. \frac{\partial^2 L}{\partial \lambda_{01}^2} \right|_{\pi_0=\pi_{11}} &= \left. \frac{\partial^2 L}{\partial \lambda_{01}^2} \right|_{\pi_0=\pi_{00}} + \lambda_{01}^{-2}; & \left. \frac{\partial^2 L}{\partial \lambda_{11}^2} \right|_{\pi_0=\pi_{11}} &= \left. \frac{\partial^2 L}{\partial \lambda_{11}^2} \right|_{\pi_0=\pi_{00}} + \delta_{11}^{-2};
 \end{aligned}$$

The other second partial derivatives  $\frac{\partial^2 L}{\partial \lambda_{ij} \partial \lambda_{ij}} \Big|_{\pi_0 = \pi_{11}}$  of  $L$  are identical to  $\frac{\partial^2 L}{\partial \lambda_{ij} \partial \lambda_{ij}} \Big|_{\pi_0 = \pi_{00}}$ .

### Appendix 3 On the maximization of the likelihood function

The maximization of the likelihood functions associated with the different  $\pi_0$  has been programmed in SAS/IML V.6.4., with the use of the Newton algorithm briefly described below.

#### Algorithm

A necessary condition for the maximum of  $L$  is that the first partial derivatives of  $\log L$

$$\frac{\partial \log L(\lambda_{ij})}{\partial \lambda_{ij}} = 0; \quad i, j = 0, 1$$

are equal to zero. This nonlinear equation system cannot be solved analytically because the likelihood function is highly nonlinear in  $\lambda_{ij}$ . Let this nonlinear equation system be represented by the vector equation

$$G(\lambda) = 0$$

where  $G$  is the gradient

$$G(\lambda) = \left[ \frac{\partial \log L}{\partial \lambda_{00}}, \frac{\partial \log L}{\partial \lambda_{01}}, \frac{\partial \log L}{\partial \lambda_{10}}, \frac{\partial \log L}{\partial \lambda_{11}} \right]$$

If  $H$  indicates the Hessian of second partial derivatives of  $\log L$  in conformity with  $G(\lambda)$  above, Newton's iterative method for solving the system is given by

$$\lambda_{n+1} = \lambda_n - H^{-1}(\lambda_n) G(\lambda_n) ;$$

The convergence criterion is set to  $< 10^{-13}$ . If  $\lambda$  is the value where convergence occurs,  $\lambda$  is taken as the maximum likelihood estimate with asymptotic covariances provided by the Hessian matrix. The program has been tested with the use of data provided by Falk (1986, p 1102).<sup>2</sup>

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<sup>2</sup> The SAS-program has also been validated by a program written in GAUSS matrix language by Dr. Claes Cassel (Stockholm School of Economics). This program does not need analytical derivatives.

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